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GEODESIC STRUCTURE IN SCHWARZSCHILD GEOMETRY WITH EXTENSIONS IN HIGHER DIMENSIONAL SPACETIMES

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GEODESIC STRUCTURE IN SCHWARZSCHILD GEOMETRY WITH EXTENSIONS IN HIGHER DIMENSIONAL SPACETIMES

A Thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Abstract

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From Birkhoff's theorem, the geometry in four spacetime dimensions outside a spherically symmetric and static, gravitating source must be given by the Schwarzschild metric. This metric therefore satisfies the Einstein vacuum equations, $R_{\mu\nu} = 0$. If the mass which gives rise to the Schwarzschild spacetime geometry is concentrated within a radius of $r = 2M$, a black hole will form.

Non-accelerating particles (freely falling) traveling through this geometry will do so along parametrized curves called geodesics, which are curved space generalizations of straight paths. These geodesics can be found by solving the geodesic equation.

In this thesis, the geodesic structure in the Schwarzschild geometry is investigated with an attempt to generalize the solution to higher dimensions.

CHAPTER 1

INTRODUCTION

1.1 Background

Since the advent of Albert Einstein's field equations published from 1914-1915 [1], which relate the geometry of the spacetime manifold to the matter content within it, a major effort has been undertaken to discover solutions which satisfy the Einstein field equations (EFE).

These solutions usually manifest themselves as line elements (metrics) which describe a particular type of geometry based on certain physical assumptions such as spherical symmetry, axisymmetric rotational symmetry, stationarity, etc.

The first instance of a closed solution to the EFE came shortly after their publication in 1916 and was due to the work of Karl Schwarzschild [2]. Schwarzschild derived a metric which described the exterior geometry of a spherically symmetric, stationary, static source, thus necessarily satisfying the vacuum form of the EFE (no sources in the manifold). In the same year, Hans Reissner generalized Schwarzschild's solution to include electrically charged sources, later independently verified by Gunnar Nordstrom, giving rise to the Reissner-Nordstrom metric [3] [4].

The next goal was to extend the solution of the stationary, static case to that of a stationary but rotating source. Due to the mathematical complexities in accomplishing such a task, it was not until 1963 that an adequate solution was found by Roy Kerr [5]. Shortly after this the charged, rotating solution was derived by Ezra Newman and named the Kerr-Newman metric [6]. The Kerr-Newman metric is the most general of the asymptotically flat, stationary black hole solutions to the

Einstein-Maxwell equations, hence it is of great importance in understanding the evolution of the geometry outside the source it describes.

Although these four metric solutions describe the geometry outside any source, more interestingly they describe exteriors to black hole solutions within the 4D manifold of general relativity (GR). It is assumed the reader has at the very least a conceptual understanding of black holes, therefore their description will be more rigorous in the chapters to follow. The nature of the event horizon will be explored in future chapters in connection with the overall geometry given by the metric.

Due to the dimensional claims made by modern theories of supergravity (string theory in particular), it is logical to ask about the nature of black holes in higher dimensions. The generalization of the Schwarzschild solution to higher dimensional spacetimes was due to Frank Tangherlini in 1963 [7]. The metric discovered by Tangherlini is a fairly straightforward extension, but contains some interesting geometric properties when one considers the dimension of the space as a parameter.

Given the framework of a particular geometric background, one might pose the question: how do things move in this spacetime? Geodesics are parametrized curves which generalize the notion of straight paths in flat space to curved spaces. Particles which are in free-fall, i.e. no generated acceleration on the part of the particle, will travel along geodesics governed by the geometry of the spacetime. One can determine these particular class of curves for a given geometry by solving the geodesic equation, to be covered in §3.6.

What follows is an exploration into the nature of geodesics in the Schwarzschild geometry culminating in a generalization to higher dimensions. Beginning in chapter 2, a brief review of special relativity is conducted in order to establish a firm foundation regarding line elements and the causal nature of the 4D world view. In chapters 3 and 4, the necessary physical principles and mathematical constructs used in GR required to understand curved spaces and higher dimensional geometry are introduced, beginning with topics in differential geometry as applied to GR. These include manifolds, tangent/cotangent spaces and their elements, tensors (metric tensor), covariant differentiation, parallel transport, and curvature as applied to geodesics. Chapter 5 is concerned with the 4D Schwarzschild geometry, with the radial geodesic equation being solved for bound orbits. Chapter 6 investigates the generalization of the Schwarzschild geometry to higher dimensions with an attempt to solve the radial geodesic equation.

1.2 Motivation

It would appear the manifold of spacetime has only four dimensions, so it might seem strange to invest so much effort in higher dimensional black hole generalizations. Besides the intrinsic interest of higher dimensional gravity, there are valid reasons for pursuing higher dimensional analogs of these objects.

Tangherlini's original motivation was based on the logic that because the laws of physics allow an extension to higher dimensions, there must exist some principle which explains the observation of only four dimensions.

String theory has emerged as a theory which combines classical GR with particle physics within the framework of higher dimensions, namely spacetime in bosonic string theory is 26-dimensional, M-theory is 11-dimensional, and superstring theory is 10-dimensional [8]. Due to this framework, exploring higher dimensional analogs of 4D objects could provide a useful laboratory for advancement of theoretical work, case in point: the first successful statistical counting on black hole entropy in string theory was performed for a 5D black hole [9].

Also, the AdS/CFT correspondence defines a relationship between the properties of a D-dimensional black hole and a $(D-1)$-dimensional quantum field theory [10].

CHAPTER 2

SPECIAL RELATIVITY REVIEW

2.1 Why a Need for Special Relativity?

The theory of Special Relativity (SR), authored by Einstein and published in 1905 [11], is actually the successful attempt of one man to consolidate ideas already in circulation within the scientific community into a consistent theory based on a minimum number of assumptions (to be discussed in § 2.2).

Using the work of Galileo, Newton, Maxwell, Lorentz, Poincare and others, Einstein formulated a theory which was invariant upon coordinate transformations relating inertial frames within the framework of classical electromagnetism, something Newtonian mechanics failed to accomplish. Maxwell's electromagnetism in a sense was a fully relativistic theory in that it predicts the speed of light in vacuum, i.e. $c = 2.99792458 \times 10^8 \frac{m}{s}$. Therefore, any theory claiming to make predictions between frames of reference better be consistent with electromagnetism.

2.2 Postulates of Special Relativity

As motivation, the null result from the 1881 Michelson-Morley experiment lead Einstein to develop two basic postulates of SR, on which the entire theory is based:

1. **Principle of Relativity:** The laws of (non-gravitational) physics assume equivalent forms in all inertial frames.

2. **Constancy of the Speed of Light:** The speed of light in vacuo assumes the same value in all inertial frames, regardless of the velocity of the observer or

source.

The first postulate is by no means new and in fact is contained within Newtonian dynamics. As a consequence, one observer cannot distinguish whether they are at rest or in uniform motion. Absolute motion does not exist, and instead motion is defined relative to another frame of reference.

The second postulate is actually contained within the first; if the speed of light were not c in all frames, then there would exist a unique, preferred frame from which to observe others. Therefore the speed of light is the same in every inertial frame, and furthermore implies space is isotropic.

2.3 Lorentz Transformations

The Galilean transformation, which Newton incorporated into his mechanics, is not a valid coordinate transformation for frames with relative velocities v comparable to the speed light. The correct transformation relating space and time coordinates in two inertial frames $[t, x, y, z]$ and $[t^{'}, x^{'}, y^{'}, z^{'}]$ which leaves the Maxwell equations invariant, the Lorentz transformation, goes as [12]

$$x^{'\alpha} = \Lambda^{\alpha}_{\beta} x^{\beta} + a^{\alpha} \tag{2.1}$$

where a^{α} and Λ^{α}_{β} are a constant vector and matrix respectively, restricted by the conditions

$$\Lambda^{\alpha}_{\gamma} \Lambda^{\beta}_{\delta} \eta_{\alpha\beta} = \eta_{\gamma\delta} \tag{2.2}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric.

As an example, a particular boost in the $x^1 = x$ direction between two frames in standard configuration reads as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{2.3}$$

with $\beta = \frac{v}{c}$, where v is the relative velocity between frames, and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ the Lorentz factor. A generalized boost in any dimension can be expressed as

$$\Lambda_{00} = \gamma \tag{2.4a}$$

$$\Lambda_{0i} = \Lambda_{i0} = -\gamma\beta_i \tag{2.4b}$$

$$\Lambda_{ij} = (\gamma - 1)\frac{\beta_i \beta_j}{\beta^2} + \delta_{ij} \tag{2.4c}$$

with Latin indices representing spatial components.

A significant feature of this linear transformation is time and space coordinates are now coupled, leaving time intervals and 3D lengths non-invariant upon this transformation. Hence time and space lose their sense of absoluteness in the Newtonian sense.

2.4 The Minkowski Line Element and 4D World View

In 3D Euclidean space, the distance squared between any two points, (x_1, y_1, z_1) and (x_2, y_2, z_2), in a particular coordinate representation is given by the Pythagorean theorem

$$s^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \quad . \tag{2.5}$$

This can also be represented for differential displacements in the coordinates as

$$ds^2 = dx^2 + dy^2 + dz^2 \quad . \tag{2.6}$$

The transformation between coordinate systems involving rotations and translations leaves Eq. (2.6) invariant. The 4D counterpart used in SR that leaves ds^2 invariant under Lorentz transformations goes as

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{2.7}$$

where the space this line element occupies is referred to as Minkowski space, after its creator Hermann Minkowski in 1908. Throughout the literature, a common label for spacetime coordinates x^μ ($\mu = 0, 1, 2, 3$) is

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z \quad . \tag{2.8}$$

Thus the line element of Eq. (2.7) can be recast as

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \quad . \tag{2.9}$$

As a precursor to Chapter 3, it will briefly be mentioned that the line element in GR can be expressed as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{2.10}$$

where Einstein convention has been adopted regarding implied summation for an index which appears as a superscript and subscript over all its values. The matrix coefficients $g_{\mu\nu}$ together form what is called the metric tensor, or more specifically $g_{\mu\nu}$ is a $(0, 2)$ tensor, or rank 2 covariant tensor. In GR, the metric coefficients are functions determined by the curvature of spacetime, but for flat space in SR they

(a) 3 Spatial Dimensions

(b) 2 Spatial Dimensions, 1 Temporal Dimension

Fig. 2.1. **(a)** Spherical wavefronts emanating from a pulse of light at the origin for different values of time, $0 < t_1 < t_2 < t_3$. **(b)** Minkowski diagram for the history of a massive particle constrained by the light cone defined by a pulse of light.

reduce to constant values $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

Points in Minkowski space are labeled events and the general evolution of these points through spacetime trace out curves, which are called worldlines. These worldlines can be parameterized in order to develop a sense of where on the curve we are, i.e. $x^\mu = x^\mu(\lambda)$ where here λ is the curve parameter. Massive particles have worldlines parameterized by what is called proper time τ, or the time measured in the inertial frame which a moving clock is considered to be at rest.

Consider for example a light source which emits a spherical wavefront at the origin ($x^0 = 0$, $x^1 = 0$, $x^2 = 0$, $x^3 = 0$). In $x^1x^2x^3$ space, successive views of the propagating wavefront at different moments in time appear as concentric spherical shells, see Fig. (2.1a). If we replace one of the spatial dimensions with the temporal one, we have a version of Minkowski space with two spatial dimensions evolving along the time axis, seen in Fig. (2.1b). This appears as a cone whose sides are defined by the speed of light, or light cone. Since nothing with mass can travel faster than the speed of light c, this creates a boundary for worldlines evolving in time.

More generally, a massive particle will lie at all times within the corresponding spherical wavefront of the light pulse and inside the cone of light. This fact is one case of three regarding the sign of the line element. A vector x^μ in Minkowski spacetime is timelike if $ds^2 < 0$, lightlike if $ds^2 = 0$, and spacelike if $ds^2 > 0$ based on the signature of the metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Fig. (2.2) encapsulates the distinction between the sign of the line element.

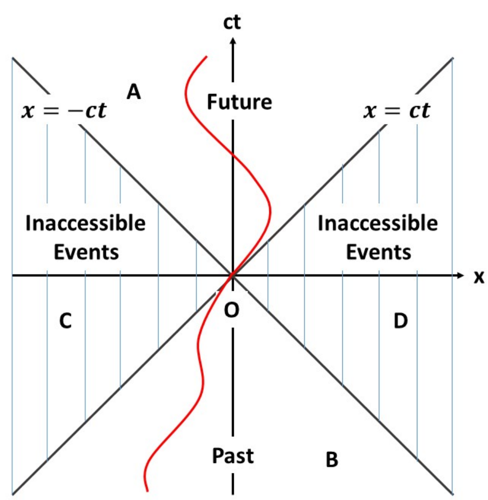


Fig. 2.2. The worldline of a particle traveling through two dimensional Minkowski space. As can be seen, the speed of light constrains the history of the particle to within the light cone. Events outside of the future/past regions are not causally linked to events within the light cone.

In region A, this is the causally allowed set of events which can be connected to the starting point ($ct = 0, x = 0$), assuming subluminal speeds. If one extends back in time, region B is just the allowed set of events which could influence the present point at the origin. Regions C and D are spacelike separated from events in A or B due to the constraint that no massive particle can exceed the speed of light. Hence, regions C and D are not causally linked to A or B.

If one wished to actually find information regarding the path a particle traversed in Minkowski space simply integrate over the line element, keeping in mind the fact ds^2 need not be positive definite. For spacelike paths the path length reads as

$$\Delta s = \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \tag{2.11}$$

for curve parameter λ. Null paths have a line element equal to zero and for timelike paths the proper time reads as

$$\Delta \tau = \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \tag{2.12}$$

where the minus sign is needed for a positive proper time. [13]

CHAPTER 3

DIFFERENTIAL GEOMETRY REVIEW

The mathematical path Einstein traversed from SR to GR required a replacement of Minkowski spacetime with a general curved spacetime, where the curvature was created by energy and momentum. Before exploring the ways in which this works, some differential geometry must be addressed concerning the mathematics of curved spaces. Firstly, manifolds will be defined, then tangent and cotangent spaces with their elements being vectors and dual vectors respectively. The nature of tensors including the metric tensor will then be explored, leading to covariant differentiation, curvature, and finally geodesics. What follows in §3.1, §3.2, §3.3 is a summarization from [14].

3.1 Manifolds

When discussing manifolds in this thesis, what is really meant are differentiable manifolds, i.e. manifolds with derivatives defined at points within the set. In order to be as technically sound as possible, let an n-dimensional manifold M be a set which satisfies the following properties:

- There exists a topology T defined on M, such that (M, T) is a topological space with the Hausdorf property.
- The topology T contains a family of open sets $S = (U_\alpha \in T : \alpha \in I)$ which covers M, such that

$$M = \underset{\alpha \in I}{\cup} U_\alpha \quad . \tag{3.1}$$

- Corresponding to each open set U_α, there is a mapping ϕ_α, such that ϕ_α is a homeomorphism from U_α onto an open subset V_α of $\mathbb{R}^n$.

- If $U_\alpha, U_\beta \in S$ where $U_\alpha \cap U_\beta \neq \emptyset$, then the composite mapping $\phi_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1}$ given by

$$\phi_{\alpha\beta} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta) \tag{3.2}$$

  is infinitely differentiable (smooth).

By requiring the manifold be a topological space, two points have a sense of "nearness". Also, since the manifold M has a family of open sets S which cover it, a local neighborhood can be defined around all points within the manifold, ensuring continuity.

The third property above guarantees the manifold is locally homeomorphic to $\mathbb{R}^n$. Given a point $p \in U_\alpha \subseteq M$, there exists a mapping (homeomorphism) ϕ_α from U_α onto an open subset $V_\alpha \in \mathbb{R}^n$, as seen in Fig. (3.1). For each $p \in U_\alpha$, $(x^1(p), x^2(p), ..., x^n(p))$ represent the coordinates of p which correspond to ϕ_α. The collection (U_α, ϕ_α) is a chart, or local coordinate system, while $((U_\alpha, \phi_\alpha) : \alpha \in I)$ is called at atlas. The last thing which should be mentioned for the third property is multiple charts can be assigned to a point $p \in M$, since p is a geometrical entity independent of its coordinate representation.

The last property imposes conditions on the transformations between charts evaluated at the same manifold point, see Fig. (3.2). In other words, the map $\phi_{\alpha\beta}$ yields a change of coordinates from one system to another such that the transformation be

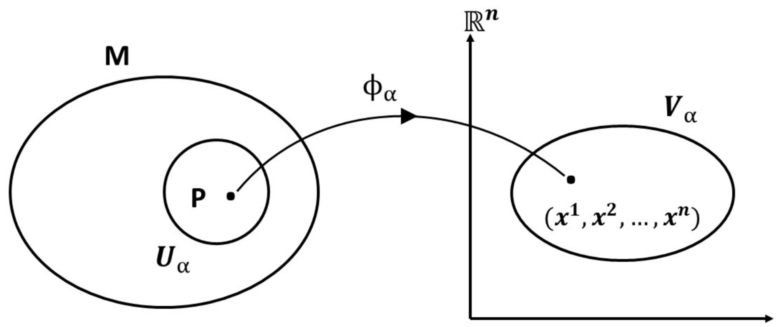


Fig. 3.1. A homeomorphism from the manifold M to $\mathbb{R}^n$.

smooth, i.e. C^∞.

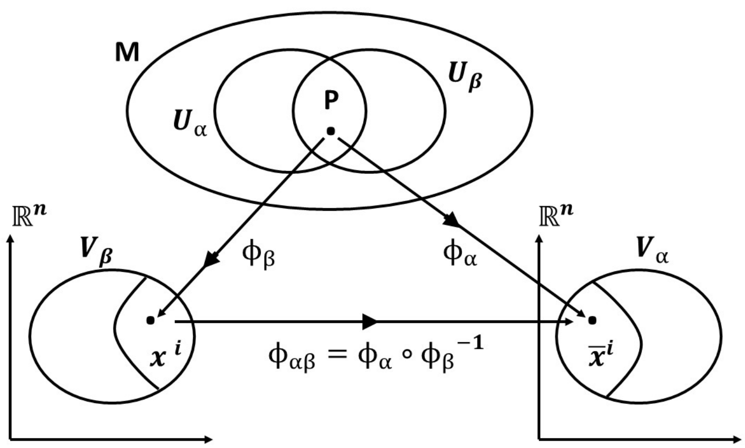


Fig. 3.2. A C^∞ change of coordinates $\bar{x}^i = \phi_{\alpha\beta}(x^i)$.

3.2 Vectors and Tangent Spaces

In order to discuss the class of geometric objects labeled vectors, the concept of a curve on a manifold M must be introduced. A curve is defined as a C^∞ map

$$\gamma : (-\epsilon, \epsilon) \to M \tag{3.3}$$

with $(-\epsilon, \epsilon) \in \mathbb{R}$. Thus a specific point in M is assigned a real number w by way of γ. This can be seen from Fig. (3.3), where the curve γ has the coordinate representation

$$x^k = \phi^k(\gamma(t)) \quad (k = 1, 2, 3, ..., n) \tag{3.4}$$

for the k-th coordinate of the point $\phi(\gamma(t)) \in \mathbb{R}^{\mathrm{n}}$.

If a function f is defined such that $f : M \to \mathbb{R}$, the function can assign a real number $f(p)$ to each $p \in M$, and using Fig. (3.3), the coordinate representation of f can be expressed as $f\phi^{-1} : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$, or

$$y = f\phi^{-1}(x^1, x^2, ..., x^n) \equiv f\phi^{-1}(\mathbf{x}) \tag{3.5}$$

where y is just a real-valued function of n variables.

For a specific $w = w_0$ mapped from $\mathbb{R}$, the tangent vector can be defined at $\gamma(w_0)$ using the directional derivative of $f(\gamma(w = w_0))$ as

$$\frac{d}{dw} f(\gamma(w))|_{w=w_0} = \frac{\partial f}{\partial x^k} \cdot \frac{dx^k}{dt}|_{w=w_0} \tag{3.6}$$

where local coordinates have been used.

Pushing further, the differential operator $\mathbf{v}$ can be defined as

$$\mathbf{v} = v^k \frac{\partial}{\partial x^k} \quad \text{where} \quad v^k \equiv \frac{dx^k}{dt}|_{w=w_0} \quad . \tag{3.7}$$

Therefore Eq. (3.6) can be written as

$$\frac{d}{dw} f(\gamma(w))|_{w=w_0} = v^k \frac{\partial f}{\partial x^k} = \mathbf{v}[f] \quad . \tag{3.8}$$

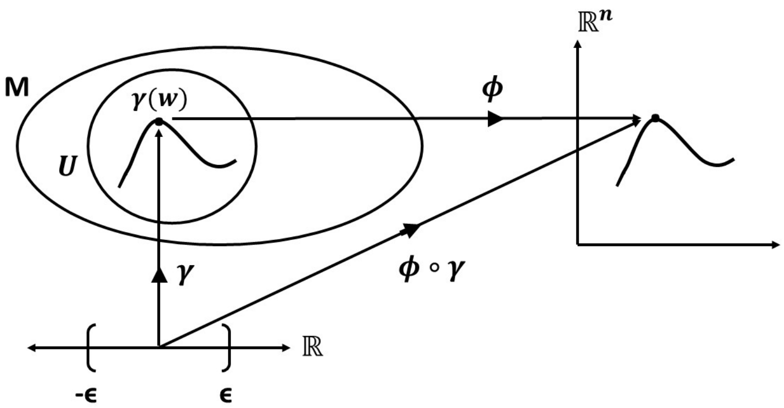


Fig. 3.3. A curve γ defined on the manifold M.

More intuitively, $\mathbf{v}[f]$ represents the rate of change of the function along the curve γ at the point $\gamma(w_0)$.

The set of all tangent vectors at $p \in M$ comprises the tangent space of M at p, called $T_p(M)$. $T_p(M)$ can be visualized as all C^∞ curves that pass through p with tangent vectors at p, seen in Fig. (3.4).

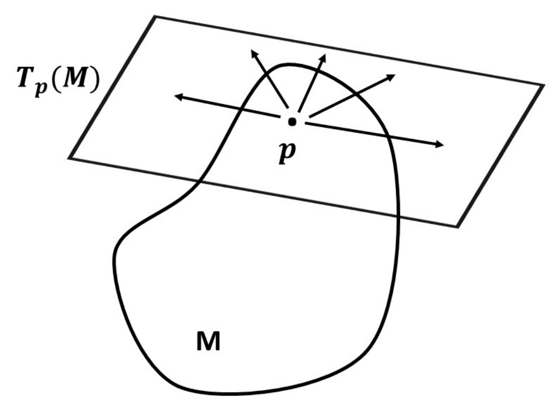


Fig. 3.4. The tangent space $T_p(M)$ at a point $p \in M$ for a general manifold M.

$T_p(M)$ has the structure of a vector space with dimension equal to that of the manifold. Vector addition and scalar multiplication are defined appropriately in regards to vector spaces.

A set of quantities $(\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n)$ can be defined such that they form a basis for $T_p(M)$, called a coordinate or holonomic basis (a local condition for a basis to be holonomic is if all mutual Lie derivatives vanish, i.e. $[e_\alpha, e_\beta] = \mathcal{L}_{e_\alpha} e_\beta = 0$). The remainder of this chapter operates under the assumption of a holonomic basis. Furthermore, vectors as elements of the tangent space can be expressed as

$$\mathbf{v} = v^k \mathbf{e}_k \tag{3.9}$$

where v^k are the components of the vector relative to this basis. Based on Eq. (3.7), these basis vectors are simply partial derivatives with respect to the k-th coordinate.

As a final note, if we have a point $p \in M$ and two coordinate charts (U, ϕ) and $(V, \bar{\phi})$ such that $p \in U \cap V$, then the components of a vector in one coordinate representation transform to the other coordinate representation as

$$\bar{v}^j = \frac{\partial \bar{x}^j}{\partial x^k} v^k \quad . \tag{3.10}$$

In a non-coordinate free way, vectors (specifically contravariant vectors) can be defined as objects whose components transform according to Eq. (3.10).

3.3 Dual Vectors and Cotangent Spaces

Consider a function f, defined on a vector space V of dimension n, such that $f : V \to \mathbb{R}$ is a linear function on V. Thus an element $\mathbf{v} \in V$ can be assigned a real number by way of f. Using Eq. (3.9) and linearity, this amounts to f acting on the basis set of $\mathbf{v}$

$$f(\mathbf{v}) = f(v^k \mathbf{e}_k) = v^k f(\mathbf{e}_k) \quad . \tag{3.11}$$

The collection of all linear functions f acting on elements of V can be denoted V^*, and it turns out this is a perfectly well suited vector space usually named the dual space or cotangent space. Elements $f \in V^*$ are named dual vectors, covectors, or 1-forms.

A basis set $(\omega^1, \omega^2, ..., \omega^n)$ can be defined for elements of the dual space in such a way that they act on the original basis vectors in V by way of

$$\boldsymbol{\omega}^n(\mathbf{e}_k) = \delta^n_k \quad . \tag{3.12}$$

A general function $f \in V^*$ can now be expressed as a linear combination of basis 1-forms as

$$f = f_n \boldsymbol{\omega}^n \tag{3.13}$$

and its action on a general vector $\mathbf{v} \in V$ reads as

$$f(\mathbf{v}) = f_n \boldsymbol{\omega}^n(v^k \mathbf{e}_k) = f_n v^k \boldsymbol{\omega}^n(\mathbf{e}_k) = f_n v^k \delta^n_k = f_n v^n \quad . \tag{3.14}$$

This action defines the inner product $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$. It is now a simple task to associate to the tangent space $T_p(M)$, with $p \in M$, a dual space $T^*_p(M)$, containing elements $\omega \in T^*_p(M)$.

Assuming the function f is C^∞, its differential turns out to be a 1-form. Consider a manifold M which has an associated coordinate chart (U, ϕ) containing the point $p \in M$, such that $x^k = \phi(p)$. The differential of an element of the dual space to this point reads as

$$df = \frac{\partial f}{\partial x^k} dx^k \quad . \tag{3.15}$$

Since df can be written as a linear combination of the set elements $\{dx^k\}$, it is easily shown that this set forms a basis for $T_p^*(M)$.

Lastly, let two coordinate charts, (U, ϕ) and $(V, \bar{\phi})$, be defined on M such that $p \in U \cap V$. Then the components of a 1-form in one coordinate representation transform to the other coordinate representation as

$$\bar{\omega}^k = \frac{\partial x^n}{\partial \bar{x}^k} \omega_n \quad . \tag{3.16}$$

In this way, 1-forms (covariant vectors) are defined such that they transform according to Eq. (3.16).

3.4 Tensors and the Metric Tensor

What follows is a summarization from [15]. In defining tensors and their properties, it is important to keep in mind the fundamental geometric objects which are being described. The previous sections introduced the concept of a geometrical point in a manifold and spaces extending from this point. This point is defined by a coordinate scheme and a particular basis having coefficients which can change depending on the coordinate system used. The manifold point however exists independent of the coordinate scheme used, thereby making any coordinate system in principle equivalent when describing the point.

Consider two sets of coordinates $(x^1, x^2, ..., x^n)$ and $(x^{'1}, x^{'2}, ..., x^{'n})$ describing a point in the manifold of dimension n such that a relation can be constructed between them by

$$x^{'i} = f^i(x^1, x^2, ..., x^n), \qquad i = 1, 2, ..., n \tag{3.17}$$

where the f^i functions are assumed to be single-valued and C^∞. Differentiating the above coordinate transformation with respect to each coordinate x^j yields a $n \times n$ transformation matrix

$$\frac{\partial x^{'i}}{\partial x^j} = \begin{bmatrix} \frac{\partial x^{'1}}{\partial x^1} & \frac{\partial x^{'1}}{\partial x^2} & \cdots & \frac{\partial x^{'1}}{\partial x^n} \\ \frac{\partial x^{'2}}{\partial x^1} & \frac{\partial x^{'2}}{\partial x^2} & \cdots & \frac{\partial x^{'2}}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^{'n}}{\partial x^1} & \frac{\partial x^{'n}}{\partial x^2} & \cdots & \frac{\partial x^{'n}}{\partial x^n} \end{bmatrix} \quad . \tag{3.18}$$

The determinant of this matrix is named the Jacobian of the coordinate transformation

$$J^{'} = \left| \frac{\partial x^{'i}}{\partial x^j} \right| \quad . \tag{3.19}$$

The differential of Eq. (3.17) reads as

$$dx^{'i} = \frac{\partial x^{'i}}{\partial x^j} dx^j \tag{3.20}$$

and inverting yields

$$dx^i = \frac{\partial x^i}{\partial x^{'j}} dx^{'j} \quad . \tag{3.21}$$

In a rather fundamental sense, tensors are multi-linear maps which take as inputs vectors and dual vectors to $\mathbb{R}$. Tensors are characterized in part by their rank, or the number of indices required to describe the array which comprises the tensor. Some common tensors are scalars (rank 0), vectors/dual vectors (rank 1), and usually anything of higher rank is deemed just a tensor. Therefore tensors are geometric objects which exist independent of the coordinate system used to describe them, while the components used to describe the tensor must be coordinate dependent.

Tensors can be defined by how they transform through two sets of coordinates. The transformation of a scalar leaves the functions of the coordinates unchanged, while a set of n quantities A^i which transforms like

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j \qquad i, j = 1, 2, ..., n \tag{3.22}$$

are called the components of a contravariant vector, or contravariant tensor of rank 1. The set of n quantities A_i are the components of a covariant vector if they transform according to

$$A'_i = \frac{\partial x^j}{\partial x'^i} A_j \qquad i, j = 1, 2, ..., n \quad . \tag{3.23}$$

For rank 2 covariant, contravariant, and mixed tensors, the above scheme can be easily extended and in general a set of quantities $A^{i_1 i_2 \ldots i_r}$ in coordinate system x^i are the components of a contravariant tensor of rank r if they transform as

$$A'^{j_1 j_2 \ldots j_r} = \frac{\partial x'^{j_1}}{\partial x^{i_1}} \frac{\partial x'^{j_2}}{\partial x^{i_2}} \ldots \frac{\partial x'^{j_r}}{\partial x^{i_r}} A^{i_1 i_2 \ldots i_r} \quad . \tag{3.24}$$

Similarly the quantities $A_{i_1 i_2 \ldots i_r}$ are the components of a covariant tensor of rank r if they transform according to

$$A'_{j_1 j_2 \ldots j_r} = \frac{\partial x^{i_1}}{\partial x'^{j_1}} \frac{\partial x^{i_2}}{\partial x'^{j_2}} \ldots \frac{\partial x^{i_r}}{\partial x'^{j_r}} A_{i_1 i_2 \ldots i_r} \tag{3.25}$$

while the quantities $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ are components of a mixed tensor of rank $(r+s)$ if they transform according to

$$A'^{l_1 l_2 \ldots l_r}_{m_1 m_2 \ldots m_s} = \left(\frac{\partial x'^{l_1}}{\partial x^{i_1}} \frac{\partial x'^{l_2}}{\partial x^{i_2}} \cdots \frac{\partial x'^{l_r}}{\partial x^{i_r}} \right) \left(\frac{\partial x^{j_1}}{\partial x'^{m_1}} \frac{\partial x^{j_2}}{\partial x'^{m_2}} \cdots \frac{\partial x^{j_s}}{\partial x'^{m_s}} \right) A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \quad . \tag{3.26}$$

Tensors also contain important symmetry/anti-symmetry properties depending on the permutation of their indices. If, for example, a contravariant or covariant tensor fulfills the condition

$$A^{ij}_k = A^{ji}_k \quad \text{or} \quad A_{ijk} = A_{jik} \tag{3.27}$$

then the tensor is symmetric with respect to the indices i and j. Also, if a contravariant or covariant tensor fulfills the condition

$$A^{ij}_k = -A^{ji}_k \quad \text{or} \quad A_{ijk} = -A_{jik} \tag{3.28}$$

then the tensor is said to be antisymmetric or skew symmetric with respect to the indices i and j.

A brief discussion regarding the algebra of tensors, specifically contraction, shall now follow. Contraction is the process by which a covariant and contravariant index is summed over to arrive at a new tensor, where the rank has been lowered by two. For example, consider the transformation of a rank two mixed tensor

$$A'^i_j = \frac{\partial x'^i}{\partial x^p} \frac{\partial x^q}{\partial x'^j} A^p_q \tag{3.29}$$

and taking $j = i$

$$A'^i_i = \frac{\partial x'^i}{\partial x^p} \frac{\partial x^q}{\partial x'^i} A^p_q = \delta^q_p A^p_q = A^p_p \tag{3.30}$$

and thus a rank two tensor has turned into a rank zero tensor, otherwise known as a scalar, specifically A^p_p is the trace of A^i_j, and invariant upon transformations.

As was briefly discussed in the SR section, specifically Eq. (2.10), the line element or metric contains the fundamental geometry describing the space under consideration. For this reason it is sometimes labeled the fundamental tensor. There are a number of equivalent ways to define the metric, for example projections of the set of basis vectors of one vector on another, but for the purposes of this work the following shall be considered its definition. If the general case is considered for a n-dimensional space, neighboring points are separated by a distance related to

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{where} \quad \mu, \nu = 1, 2, ..., n \tag{3.31}$$

with the $g_{\mu\nu}$'s being a n^2 number of functions such that the determinant $g = |g_{\mu\nu}| \neq 0$. The metric is a rank 2 symmetric, covariant tensor therefore its transformation reads as

$$g^{'}_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu} \tag{3.32}$$

with the property that

$$g_{ij} g^{ik} = \delta^k_j \quad . \tag{3.33}$$

It is interesting to note that $g = |g_{\mu\nu}|$ does not transform like a rank 0 tensor due to the fact that under transformations of g a factor of the Jacobian squared rears up. In general, a quantity $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ is a relative tensor of density W if it transforms according to

$$A'^{l_1 l_2 \ldots l_r}_{m_1 m_2 \ldots m_s} = \left|\frac{\partial x}{\partial x'}\right|^W \left(\frac{\partial x'^{l_1}}{\partial x^{i_1}} \frac{\partial x'^{l_2}}{\partial x^{i_2}} \cdots \frac{\partial x'^{l_r}}{\partial x^{i_r}}\right) \left(\frac{\partial x^{j_1}}{\partial x'^{m_1}} \frac{\partial x^{j_2}}{\partial x'^{m_2}} \cdots \frac{\partial x^{j_s}}{\partial x'^{m_s}}\right) A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \tag{3.34}$$

where the constant W is the weight of the tensor density.

The covariant and contravariant form of the metric tensor can be used for raising and lowering indices of tensors, which results in another tensor, and in general

$$T^{\dots i_r \dots}_{\dots k_s \dots} g_{i_r m} g^{k_s n} = T^{\dots n \dots}_{\dots m \dots} \quad . \tag{3.35}$$

3.5 Covariant Derivative and Connections

The notion of the partial derivative is assumed to be known to the reader as it pertains to objects, specifically functions, in flat space. However, it turns out a partial derivative operator acting on a vector or tensor does not return a tensorial object in curved space, and since it is known tensors are geometric objects which exist independently of the coordinate systems used to describe them, this presents a problem for using tensors to represent physical systems. What follows is a discussion regarding corrections to the derivative notion such that a tensor is returned from the derivative operation and the implications from that.

The derivative operator, named the covariant derivative, in a particular coordinate direction μ is given by the symbol ∇_μ and the act of its operation on a vector $\mathbf{V} = V^\nu e_\nu$ goes as [13]

$$\begin{aligned} \nabla_\mu \mathbf{V} &= \nabla_\mu (V^\nu e_\nu) \\ &= (\nabla_\mu V^\nu) e_\nu + V^\nu (\nabla_\mu e_\nu) \\ &= (\partial_\mu V^\nu) e_\nu + V^\nu (\Gamma^\alpha_{\nu\mu} e_\alpha) \quad . \\ &= (\partial_\mu V^\nu) e_\nu + V^\alpha (\Gamma^\nu_{\alpha\mu} e_\nu) \\ &= [\partial_\mu V^\nu + \Gamma^\nu_{\alpha\mu} V^\alpha] e_\nu \end{aligned} \tag{3.36}$$

So what has happened in the calculation above? The covariant derivative is assumed to satisfy the linearity and Leibniz rule properties leading to the second line. In the third line, we equated the covariant derivative of the basis vectors to a connection expanded in the e_α basis. In this sense, the connection is a $n \times n$ matrix (where n is the manifold dimensionality) for each coordinate direction μ which encapsulates how basis vectors change across a space. To finish, the dummy indices α and ν were interchanged so the basis set could be factored out. It is now obvious all the action happens on the coefficients of the vector field in a specific basis.

It is not so difficult a task to show the transformation of the connection from one set of coordinates to another goes as

$$\Gamma^{\nu'}_{\lambda'\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\lambda\mu} - \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} \quad . \tag{3.37}$$

This transformation law almost has the tensorial transformation requirement except for the pesky second derivative term, which implies the connection is tensorial for linear transformations only. But take heart, the connection was constructed in such a way as to make Eq. (3.36) tensorial under coordinate transformations, so it matters not the Γ's are not a tensor.

A similar situation can be analyzed for the covariant derivative of a dual vector $\boldsymbol{\omega} = \omega_\nu e^\nu$ to yield the result (after imposing the two conditions that the covariant derivative commutes with index contractions and reduces to the partial derivative on scalars)

$$\nabla_\mu \boldsymbol{\omega} = [\partial_\mu \omega^\nu - \Gamma^\lambda_{\nu\mu}\omega_\lambda]e^\lambda \quad . \tag{3.38}$$

The connection coefficients contain all the information necessary to take the covariant derivative of a tensor of arbitrary rank, which can be expressed as

$$\begin{aligned}\nabla_\sigma T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} = \quad & \partial_\sigma T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} \\ & + \Gamma^{\mu_1}_{\lambda\sigma} T^{\lambda\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_2}_{\lambda\sigma} T^{\mu_1\lambda\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} + \ldots \qquad . \\ & - \Gamma^{\lambda}_{\nu^1\sigma} T^{\mu_1\mu_2\ldots\mu_k}_{\lambda\nu_2\ldots\nu_l} - \Gamma^{\lambda}_{\nu^2\sigma} T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\lambda\ldots\nu_l} - \ldots\end{aligned} \tag{3.39}$$

As it turns out, the difference between two connections is in fact a tensor and if a connection is specified by $\Gamma^\lambda_{\nu\mu}$ another connection can be formed by simply permuting the covariant indices. Thus for any given connection a special tensor named the torsion tensor can be associated with it given by

$$T^\lambda_{\nu\mu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} = 2\Gamma^\lambda_{[\nu\mu]} \quad . \tag{3.40}$$

Therefore a symmetric connection necessarily implies a torsion-free one.

A unique connection can be determined on a manifold with a metric $g_{\mu\nu}$ assuming it is torsion-free and has metric compatibility, i.e. $\nabla_\rho g_{\mu\nu} = 0$. Thus the connection can be shown to be

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \quad . \tag{3.41}$$

Eq. (3.41) is the connection used in general relativity theory and is sometimes referred to as the Christoffel connection, Levi-Civita connection, or Riemannian connection.

3.6 Parallel Transport, Geodesic Equation, and Killing Vectors

In flat space, the notion of vectors being elements of a tangent space at a point in the manifold was masked because when computing vector sums, projections, etc., one could arbitrarily move these vectors around to performs operations. In curved space, however, this idea is not so simple.

Parallel transport is the concept of moving a vector along a curve while keeping the vector constant. This implicitly utilizes a connection for the space being traversed as will be shown soon. As it turns out, the result of parallel transporting a vector depends on the path taken from one point to another. In other words, there is no path-independent way to uniquely move a vector from one tangent space to another.

Given a curve $x^\mu(\lambda)$, a tensor A of any rank along this curve to be considered constant must satisfy $\frac{dA}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial A}{\partial x^\mu}$ and from this the definition of the covariant derivative taken along a path goes as [13]

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu \quad . \tag{3.42}$$

Thus, in general the equation of parallel transport for a tensor A along $x^\mu(\lambda)$ must be

$$\left(\frac{D}{d\lambda}A\right)^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma A^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} = 0 \tag{3.43}$$

and for a vector V^μ the parallel transport equation goes as

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0 \quad . \tag{3.44}$$

Hence it can be seen the dependency of the parallel transport process on the connection defined on the manifold is made manifest. As a last note on parallel transport, if the connection is metric compatible, the metric is always parallel transported, therefore implying the preservation of vector norms, orthogonality, etc., as the space is traversed.

A geodesic can be thought of roughly as the curved surface version of a "straight line" in flat space. Slightly more rigorously a geodesic can be defined as a path which parallel transports its own tangent vector. Immediately using this definition for a general path $x^\mu(\lambda)$, its tangent vector reads as $\frac{d}{d\lambda}x^\mu(\lambda)$, and utilizing Eq. (3.44) the geodesic equation is arrived at eerily easily

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \quad . \tag{3.45}$$

As a quick test, if Minkowski space or Euclidean space is considered, the connection coefficients would all equate to zero, yielding $\frac{d^2x^\mu}{d\lambda^2} = 0$, the equation for a straight line, which is exactly the path one would expect a free particle to travel. There is indeed another way to arrive at the geodesic equation which involves varying the line element in Lorentzian spacetime to generate the extrema of the functional under the integral sign. This however is more complicated than is needed and can be found in any literature search.

In regards to quantities undergoing coordinate transformations, an interesting subset of cases is when the quantity remains invariant under the transformation, specifically the metric tensor. The metric is form invariant under a transformation $x \rightarrow x^{'}$ when the transformed metric $g^{'}_{\mu\nu}(x^{'})$ is the same function of its argument $x^{'\mu}$ as the original metric $g_{\mu\nu}$ was of its argument x^μ [12], i.e.

$$g^{'}_{\mu\nu}(x) = g_{\mu\nu}(x) \quad . \tag{3.46}$$

At any point the transformation of the metric reads as

$$g^{'}_{\mu\nu}(x^{'}) = \frac{\partial x^\rho}{\partial x^{'\mu}}\frac{\partial x^\sigma}{\partial x^{'\nu}}g_{\rho\sigma}(x) \tag{3.47}$$

with inverse found by switching primes. When Eq. (3.46) is valid, a replacement of $g^{'}_{\rho\sigma}(x^{'})$ with $g_{\rho\sigma}(x^{'})$ leads to the requirement for form invariance

$$g_{\mu\nu}(x) = \frac{\partial x^{'\rho}}{\partial x^\mu}\frac{\partial x^{'\sigma}}{\partial x^\nu}g_{\rho\sigma}(x^{'}) \quad . \tag{3.48}$$

Coordinate transformations which satisfy the above relationship are called isometries.

Consider the following transformation

$$x^i \to x^{'i} = x^i + \epsilon \xi^i(x) \tag{3.49}$$

where here ϵ is a small parameter and $\xi^i(x)$ is a smooth vector field. This transformation is known as an infinitesimal mapping which connects different points in spacetime using the same coordinate system.

If one allows a general tensor field $T(x)$ to be defined on the manifold in question, then one can construct the Lie derivative as [15]

$$\mathcal{L}_\xi T(x) = \lim_{\epsilon \to 0} \frac{T(x) - T^{'}(x)}{\epsilon} \quad . \tag{3.50}$$

The Lie derivative evaluates the change in a tensor field in the direction of a separate vector field. This change is an invariant quantity, thus does not have a result dependent upon a coordinate representation.

An important case for what follows is the Lie derivative for the covariant rank 2 metric tensor g_{ij} which can be shown to be

$$\mathcal{L}_\xi g_{ij} = \xi^m g_{ij;m} + g_{il}\xi^l_{;j} + g_{kj}\xi^k_{;l} \quad . \tag{3.51}$$

Since the geometry of spacetime is characterized by the metric, this motivates the question as to whether the metric is invariant under the infinitesimal coordinate transformation. There exists an isometric mapping of spacetime of the form in Eq. (3.49) if the Lie derivative of the metric is zero [16]

$$\mathcal{L}_\xi g_{ij} = 0 \quad . \tag{3.52}$$

From Eq. (3.52), this equates to

$$\xi^m g_{ij;m} + g_{il}\xi^l_{;j} + g_{kj}\xi^k_{;l} = 0 \quad . \tag{3.53}$$

Since in general relativity metric compatibility is assumed, this leads to Eq. (3.54) being represented as

$$\begin{aligned}\mathcal{L}_\xi g_{ij} &= g_{il}\xi^l_{;j} + g_{kj}\xi^k_{;l} \\ &= \xi_{i;j} + \xi_{j;l} \\ &= 2\xi_{(i;j)} \\ &= 0 \quad .\end{aligned} \tag{3.54}$$

This result is known as the Killing equation, and the vectors $\xi^i(x)$ which satisfy this differential equation are Killing vectors, which characterize the symmetry properties in an invariant way. An often convenient way of identifying Killing vectors is to consider a metric which is cast in a coordinate system such that the components of the metric are independent of a certain coordinate. The basis vector for that particular cyclic coordinate is therefore a Killing vector.

As an important and useful application, which utilizes the geodesic equation and Killing's equation, consider a geodesic curve $x^\mu(\lambda)$ with tangent vector $u^\mu = \frac{d}{d\lambda}x^\mu$. If there exists a Killing vector field K^ν then the quantity $K_\mu u^\mu =$ constant.

Before the close of this section, it is beneficial to discuss a couple relationships between the Killing equation and gravitational fields. One says a gravitational field is stationary if it admits a time-like Killing vector field $\xi^i(x)$, i.e. $\xi^2 = g_{ij}\xi^i\xi^j < 0$. As an extension, if one is dealing with a stationary field and if the killing vector(s) are orthogonal to a family of hypersurfaces then the field is said to be static. In the section on 4D black holes, the Schwarzschild geometry has both these properties.

3.7 Curvature

What follows is a summarization from [12] [13] [17]. In order to flesh out the idea of curvature and derive an expression representing it, consider the commutator of two covariant derivatives. This idea is in a way connected with the notion of parallel transport discussed earlier in that the covariant derivative of a tensor in a particular direction measures the change relative to what it would have been if parallel transported (since the covariant derivative along a path of parallel transport returns zero). The commutator of two covariant derivatives, seen as

$$[\nabla_\mu, \nabla_\nu] = \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \tag{3.55}$$

encapsulates the difference in parallel transporting a tensor one way as compared to the reverse, see Fig. (3.5).

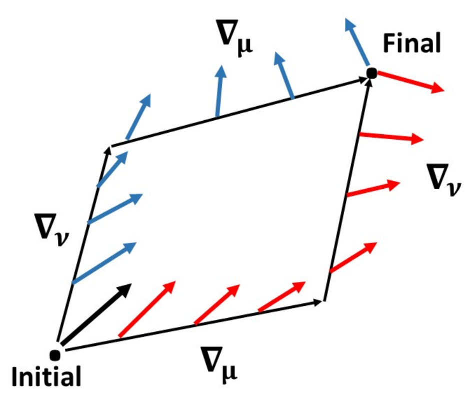


Fig. 3.5. Difference in successive covariant differentiations of a tensor along two separate paths.

Specifically, consider the components of a vector $\mathbf{V}$, namely V^ρ, and observe the act of operating on it with successive covariant derivatives

$$[\nabla_\mu, \nabla_\nu]V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho \quad . \tag{3.56}$$

One only really needs to calculate $\nabla_\mu \nabla_\nu V^\rho$ because the other term in Eq. (3.57) is the same except the indices μ and ν are flipped. Thus proceeding in this fashion

$$\begin{aligned}
\nabla_\mu \nabla_\nu V^\rho &= \partial_\mu(\nabla_\nu V^\rho) - \Gamma^\lambda_{\nu\mu}(\nabla_\lambda V^\rho) + \Gamma^\rho_{\alpha\mu}(\nabla_\nu V^\alpha) \\
&= \partial_\mu(\partial_\nu V^\rho + \Gamma^\rho_{\alpha\nu} V^\alpha) - \Gamma^\lambda_{\nu\mu}(\partial_\lambda V^\rho + \Gamma^\rho_{\alpha\lambda} V^\alpha) + \Gamma^\rho_{\alpha\mu}(\partial_\nu V^\alpha + \Gamma^\alpha_{\beta\nu} V^\beta) \\
&= \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\alpha\nu}) V^\alpha + \Gamma^\rho_{\alpha\nu}(\partial_\mu V^\alpha) - \Gamma^\lambda_{\nu\mu} \partial_\lambda V^\rho - \Gamma^\lambda_{\nu\mu} \Gamma^\rho_{\alpha\lambda} V^\alpha \\
&\qquad + \Gamma^\rho_{\alpha\mu} \partial_\nu V^\alpha + \Gamma^\rho_{\alpha\mu} \Gamma^\alpha_{\beta\nu} V^\beta \quad .
\end{aligned} \tag{3.57}$$

Therefore the other term must be

$$\begin{aligned}
\nabla_\nu \nabla_\mu V^\rho = \partial_\nu \partial_\mu V^\rho + (\partial_\nu \Gamma^\rho_{\alpha\mu}) V^\alpha + \Gamma^\rho_{\alpha\mu}(\partial_\nu V^\alpha) - \Gamma^\lambda_{\mu\nu} \partial_\lambda V^\rho \\
- \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\alpha\lambda} V^\alpha + \Gamma^\rho_{\alpha\nu} \partial_\mu V^\alpha + \Gamma^\rho_{\alpha\nu} \Gamma^\alpha_{\beta\mu} V^\beta
\end{aligned} \tag{3.58}$$

and the commutator becomes

$$\begin{aligned}
\big[\nabla_\mu, \nabla_\nu\big] V^\rho = (\partial_\mu \Gamma^\rho_{\alpha\nu} - \partial_\nu \Gamma^\rho_{\alpha\mu}) V^\alpha + (\Gamma^\rho_{\alpha\mu} \Gamma^\alpha_{\beta\nu} - \Gamma^\rho_{\alpha\nu} \Gamma^\alpha_{\beta\mu}) V^\beta \\
+ (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu})(\partial_\lambda V^\rho + \Gamma^\rho_{\alpha\lambda} V^\alpha) \quad .
\end{aligned} \tag{3.59}$$

Relabeling α to β on the first term the commutator can now be shown to be

$$\begin{aligned}
\big[\nabla_\mu, \nabla_\nu\big] V^\rho &= (\partial_\mu \Gamma^\rho_{\beta\nu} - \partial_\nu \Gamma^\rho_{\beta\mu} + \Gamma^\rho_{\alpha\mu} \Gamma^\alpha_{\beta\nu} - \Gamma^\rho_{\alpha\nu} \Gamma^\alpha_{\beta\mu}) V^\beta + (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu})(\partial_\lambda V^\rho + \Gamma^\rho_{\alpha\lambda} V^\alpha) \\
&= (\partial_\mu \Gamma^\rho_{\beta\nu} - \partial_\nu \Gamma^\rho_{\beta\mu} + \Gamma^\rho_{\alpha\mu} \Gamma^\alpha_{\beta\nu} - \Gamma^\rho_{\alpha\nu} \Gamma^\alpha_{\beta\mu}) V^\beta + 2\Gamma^\lambda_{[\mu\nu]} \nabla_\lambda V^\rho \quad .
\end{aligned} \tag{3.60}$$

If the following definitions are made

$$R^{\rho}_{\beta\mu\nu} = \partial_\mu \Gamma^{\rho}_{\beta\nu} - \partial_\nu \Gamma^{\rho}_{\beta\mu} + \Gamma^{\rho}_{\alpha\mu}\Gamma^{\alpha}_{\beta\nu} - \Gamma^{\rho}_{\alpha\nu}\Gamma^{\alpha}_{\beta\mu} \tag{3.61a}$$

$$T^{\lambda}_{\mu\nu} = 2\Gamma^{\lambda}_{[\mu\nu]} \tag{3.61b}$$

where $R^{\rho}_{\beta\mu\nu}$ is the Riemann curvature tensor and $T^{\lambda}_{\mu\nu}$ is the torsion tensor, then the commutator of successive covariant derivatives can be expressed as

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^{\rho}_{\beta\mu\nu}V^\beta + T^{\lambda}_{\mu\nu}\nabla_\lambda V^\rho \quad . \tag{3.62}$$

The Riemann tensor measures the part of the commutator of successive covariant derivatives which is proportional to the vector field, and the torsion tensor the part which is proportional to the covariant derivative of the vector field.

The action of the commutator of covariant derivatives on tensors of arbitrary rank can be easily extended as

$$\begin{aligned} X^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} = &- T^{\lambda}_{\rho\sigma}\nabla_\lambda X^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} \\ &+ R^{\mu_1}_{\lambda\rho\sigma} X^{\lambda\mu_2 \ldots \mu_k}_{\nu_1 \ldots \nu_l} + R^{\mu_2}_{\lambda\rho\sigma} X^{\mu_1 \lambda \ldots \mu_k}_{\nu_1 \ldots \nu_l} + \ldots \quad . \\ &- R^{\lambda}_{\nu_1\rho\sigma} X^{\mu_1 \ldots \mu_k}_{\lambda\nu_2 \ldots \nu_l} - R^{\lambda}_{\nu_2\rho\sigma} X^{\mu_1 \ldots \mu_k}_{\nu_1 \lambda \ldots \nu_l} - \ldots \end{aligned} \tag{3.63}$$

In GR, the Christoffel connection is what is considered which yields symmetric connections, therefore the torsion is automatically made zero. Based on the construction of the Riemann curvature tensor, it can be understood as a system of second order partial differential equations with respect to the metric. Thus if, in a particular coordinate system, the metric has constant components, then the Riemann curvature tensor will vanish. In other words, if a coordinate system is found such that $\partial_\sigma g_{\mu\nu} = 0$, then $\Gamma^{\rho}_{\mu\nu} = 0$ and $\partial_\sigma \Gamma^{\rho}_{\mu\nu} = 0$, therefore $R^{\rho}_{\sigma\mu\nu} = 0$. This is a tensor equation and as such its result is independent of coordinate representation.

One might naively assume the Riemann tensor has n^4 components for an n-dimensional manifold, but in fact the Riemann tensor is antisymmetric in its last two indices

$$R^{\rho}_{\sigma\mu\nu} = -R^{\rho}_{\sigma\nu\mu} \tag{3.64}$$

which reduces the number of independent components to $\frac{n(n-1)}{2}$ values in the last two indices leaving a total of $\frac{n^3(n-1)}{2}$ independent components. This is still general however and has not made any use of a metric on the manifold. If the Christoffel connection is considered, the number of independent components reduces even further. To see this, lower the contravariant index on the Riemann tensor

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^{\lambda}_{\sigma\mu\nu} \tag{3.65}$$

and assume Riemann normal coordinates such that the first derivatives of the metric vanish at some point p on the manifold. The only surviving terms in the Riemann tensor go as

$$\begin{aligned} R_{\rho\sigma\mu\nu} &= g_{\rho\lambda}(\partial_\mu \Gamma^{\lambda}_{\nu\sigma} - \partial_\nu \Gamma^{\lambda}_{\mu\sigma}) \\ &= \frac{1}{2}(\partial_\mu\partial_\sigma g_{\rho\nu} - \partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\nu\partial_\sigma g_{\rho\mu} + \partial_\nu\partial_\rho g_{\mu\sigma}) \quad . \end{aligned} \tag{3.66}$$

Certain symmetries which can be seen from Eq. (3.61) are:

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \tag{3.67}$$

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \tag{3.68}$$

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0 \tag{3.69}$$

and taking these into account it can be shown the Riemann tensor has $\frac{n^2(n^2-1)}{12}$ independent components and for 4D spacetime this works out to be 20 independent components. The above analysis was done in a special coordinate system but due to the tensorial nature of $R_{\rho\sigma\mu\nu}$ it holds true in any frame.

Another identity comes in the form of the covariant derivative of the Riemann tensor in Riemann normal coordinates

$$\begin{aligned}\nabla_\lambda R_{\rho\sigma\mu\nu} &= \partial_\lambda R_{\rho\sigma\mu\nu} \\ &= \frac{1}{2}\partial_\lambda(\partial_\mu\partial_\sigma g_{\rho\nu} - \partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\nu\partial_\sigma g_{\rho\mu} + \partial_\nu\partial_\rho g_{\mu\sigma})\end{aligned} \tag{3.70}$$

and if the sum of cyclic permutations in the first three indices is considered it can be shown

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0 \quad . \tag{3.71}$$

The last two identities are known as the Bianchi identities.

Finding the trace of the Riemann tensor equates to contracting the first and third indices yielding

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \tag{3.72}$$

and this result is known as the Ricci tensor. Note for the Christoffel connection, this is the only independent contraction, which is also symmetric due to the symmetry of multiple partial derivatives within the Riemann tensor. What is known as the Ricci scalar can be computed with help from the metric as

$$R = R^\mu_\mu = g^{\mu\nu}R_{\mu\nu} \quad . \tag{3.73}$$

If Eq. (3.72) in contracted twice

$$
\begin{aligned}
0 &= g^{\nu\sigma} g^{\mu\lambda} (\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu}) \\
&= \nabla^\mu R_{\rho\mu} - \nabla_\rho R + \nabla^\sigma R_{\lambda\rho\mu\nu}
\end{aligned}
\tag{3.74}
$$

and the dummy index ν is interchanged for μ one can arrive at

$$
\begin{aligned}
\nabla^\mu R_{\rho\mu} &= \frac{1}{2} \nabla_\rho R \\
&= \frac{1}{2} \nabla^\mu g_{\mu\rho} R \quad .
\end{aligned}
\tag{3.75}
$$

If Eq. (3.76) is rearranged one can find

$$
\nabla^\mu G_{\mu\nu} = 0 \tag{3.76}
$$

where $G_{\mu\nu}$ is known as the Einstein tensor and can be expressed as

$$
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad . \tag{3.77}
$$

This equation is quite significant and will play an important role in the Einstein field equations which describe the interaction between the energy-momentum content within the 4D spacetime manifold and the curvature of spacetime itself.

The final topic which shall be covered in this section is that of geodesic deviation. In flat space, initially parallel lines will remain so if extended infinitely far, as given by Euclid in his *Elements* as the parallel postulate. In curved space this is not the case. Instead consider a family of geodesics $\gamma_s(t)$, where s can assume any real value and Affine parameter t. These curves define a smooth 2D surface in a manifold of arbitrary dimension with coordinates s and t such that the geodesics do not intersect.

Consider two vector fields: the tangent vectors to geodesics

$$
T^\mu = \frac{\partial x^\mu}{\partial t} \tag{3.78}
$$

and deviation vectors which point from one geodesic to a neighboring one, as seen in Fig. (3.6).

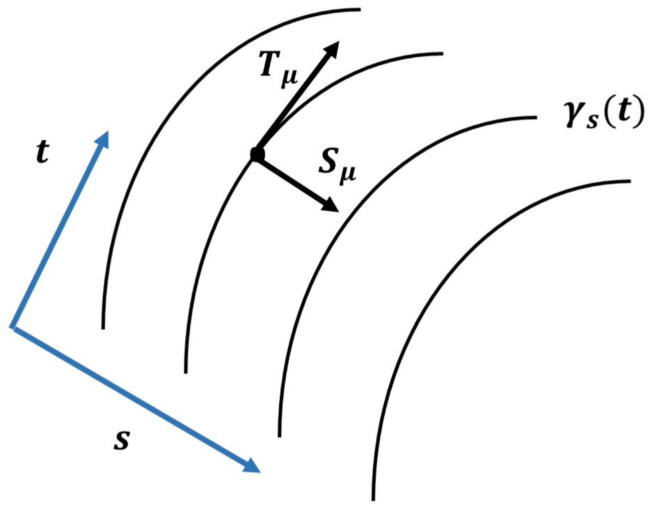


Fig. 3.6. Representation of geodesic deviation in s and t basis for the family of geodesics $\gamma_s(t)$.

$$S^\mu = \frac{\partial x^\mu}{\partial s} \quad . \tag{3.79}$$

Based on this construct, the rate of change of these geodesics goes as

$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu \tag{3.80}$$

where here T^μ are the coefficients of the vector for which the covariant derivative is being taken in the direction of. The acceleration of the separation between neighboring geodesics goes as

$$\begin{aligned} a^\mu = (\nabla_T V)^\mu &= T^\rho \nabla_\rho V^\mu \\ &= T^\rho \nabla_\rho (T^\rho \nabla_\rho S^\mu) \end{aligned} \tag{3.81}$$

and utilizing the fact that the S and T bases commute coupled with torsion vanishing given as

$$S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu \tag{3.82}$$

The acceleration becomes

$$\begin{aligned} a^\mu &= T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu) \\ &= (T^\rho \nabla_\rho S^\sigma) \nabla_\sigma T^\mu + T^\rho S^\sigma (\nabla_\rho \nabla_\sigma T^\mu) \\ &= (S^\rho \nabla_\rho T^\sigma) \nabla_\sigma T^\mu + T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu + R^\mu_{\nu\rho\sigma} T^\nu) \\ &= (S^\rho \nabla_\rho T^\sigma) \nabla_\sigma T^\mu + S^\sigma \nabla_\sigma (T^\rho \nabla_\rho T^\mu) - (S^\sigma \nabla_\sigma T^\rho) \nabla_\rho T^\mu + R^\mu_{\nu\rho\sigma} T^\nu T^\rho S^\sigma \\ &= R^\mu_{\nu\rho\sigma} T^\nu T^\rho S^\sigma \quad . \end{aligned} \tag{3.83}$$

This last result encapsulates the notion that the relative acceleration between two geodesics is proportional to the curvature, known as the geodesics deviation equation. Physically, this phenomenon is interpreted as gravitational tidal forces.

CHAPTER 4

GRAVITATION

What follows is a summarization from [13] [17]. The presence of energy and momentum in the 4D spacetime manifold causes the spacetime to curve, which manifests itself as gravity, that can influence the paths of other particles. In order to take Poisson's equation for gravity

$$\nabla^2 \Phi = 4\pi G \rho \tag{4.1}$$

with Φ the gravitational potential, ρ the mass density, and generalize it, which is essentially the method Einstein used to arrive at his field equations, certain principles must be addressed.

The first is the Weak Equivalence Principle (WEP). The WEP states that inertial mass and gravitational mass of any object are one in the same.

$$\mathbf{F} = m_i \mathbf{a} \qquad \longleftrightarrow \qquad \mathbf{F}_g = -m_g \nabla \Phi \quad . \tag{4.2}$$

At first glance, this might seem strange given m_g has a character specific to the gravitational force, while m_i encapsulates an object's resistance to change in motion. However, Galileo showed the acceleration of different objects is the same, and independent of the mass, for objects in a gravitational field, which leads to

$$m_i = m_g \quad . \tag{4.3}$$

Thus an immediate consequence of this is the universality of the behavior of free-falling particles, i.e. the acceleration experienced is mass independent

$$\mathbf{a} = -\nabla\Phi \quad . \tag{4.4}$$

Another way to characterize the WEP is to state, in a small enough region of spacetime where gravitational tidal effects are negligible, the laws of freely-falling particles are equivalent between a gravitational field and a uniformly accelerated reference frame.

A generalization of the WEP comes in the form of the Einstein Equivalence Principle (EEP) which states that in a small enough region of spacetime, where curvature is negligible, the laws of physics reduce to those of SR, which has a special family of reference frames that are inertial (unaccelerated). This statement is alluding to the notion that gravity is simply curved spacetime and furthermore the concept of the acceleration due to gravity is something which cannot be reliably defined. In the context of GR, it is advantageous to define an unaccelerated particle as freely-falling. This directly implies gravity is not force in the traditional sense given that a freely-falling particle moving in a gravitational field is unaccelerated. To go even further, this also implies inertial frames can only be defined locally since if the frame were to be extended indefinitely at some point particles would look as if they were accelerating with respect to the local frame.

The fact that sufficiently small regions of spacetime can be represented by the laws of SR and local inertial frames can be constructed corresponds to the establishment of Riemann normal coordinates at any one point on a manifold. The failure to be able to compare the dynamics of particles in sufficiently separated inertial frames corresponds to the path-dependence of parallel transport on a curved manifold. Therefore, it should be clear that spacetime structure can be represented as a curved manifold and gravity is just the manifestation of this curvature.

With the EEP in mind, the laws of physics in sufficiently small regions of space-time reduce to those of SR. In other words, using Riemann normal coordinates at some point P, the equations describing particle motion are the same as would be in flat space. Geodesics for these particles can be seen as

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \quad \rightarrow \quad \frac{d^2x^\mu}{d\lambda^2} = 0 \tag{4.5}$$

due to the fact the connection vanishes at the point P in Riemann normal coordinates.

In order to fully arrive at the Einstein field equations, one must observe the weak field limit of Newtonian gravity; slow speeds compared to light, the gravitational field is just a perturbation of flat space, and the field is static. If the particle is moving slow compared to the speed of light the following condition can be imposed

$$\frac{dx^i}{d\tau} << \frac{dt}{d\tau} \quad . \tag{4.6}$$

Therefore, the only non-negligible term in the sum for the geodesic equation is

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 \tag{4.7}$$

where here the Christoffel symbol Γ^μ_{00}, using Eq. (3.41), can be expressed as

$$\begin{aligned} \Gamma^\mu_{00} &= \frac{1}{2}g^{\mu\lambda}(\partial_0 g_{0\lambda} + \partial_0 g_{\lambda 0} - \partial_\lambda g_{00}) \\ &= -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00} \end{aligned} \tag{4.8}$$

due to the condition of the field being static. Since a weak field is being assumed, the metric can be decomposed into the Minkowski metric plus a small perturbation term

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4.9}$$

where $|h_{\mu\nu}| << 1$. It can also be easily shown

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{4.10}$$

so therefore Eq. (4.8) becomes

$$\Gamma^{\mu}_{00} = -\frac{1}{2}\eta^{\mu\lambda}\partial_{\lambda}h_{00} \quad . \tag{4.11}$$

The geodesic equation thus becomes

$$\frac{d^2x^{\mu}}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_{\lambda}h_{00}\left(\frac{dt}{d\tau}\right)^2 \quad . \tag{4.12}$$

If the spacelike components are examined, we have for the geodesic equation

$$\frac{d^2x^{i}}{d\tau^2} = \frac{1}{2}\partial_{i}h_{00}\left(\frac{dt}{d\tau}\right)^2 \tag{4.13}$$

and switching the differentiating coordinate from proper time to coordinate time

$$\frac{d^2x^{i}}{dt^2} = \frac{1}{2}\partial_{i}h_{00} \quad . \tag{4.14}$$

If the above expression is compared to Eq. (4.4), they can be equated if the identification

$$h_{00} = -2\Phi \tag{4.15}$$

is made, which correctly yields the weak field Newtonian metric

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)[dx^2 + dy^2 + dz^2] \quad . \tag{4.16}$$

Thus the curvature of spacetime is all that is needed to describe gravity in the Newtonian limit.

Now returning to the generalization of Eq. (4.1) to arrive at the Einstein field equations, it can be seen we need a relationship between tensors, given their coordinate independent nature. It can be easily argued the generalization of the mass density ρ is just the stress-energy tensor $T_{\mu\nu}$. Something which should be expected is an expression, using the weak field metric Eq. (4.16), that reduces to

$$\nabla^2 h_{00} = -8\pi G T_{00} \quad . \tag{4.17}$$

The gravitational potential Φ should be replaced by the metric since the metric is the field quantity in question. In terms of the differential operator acting on the potential, it is known the Riemann curvature tensor is constructed from second derivatives of the metric, and in order to match indices, it can be contracted to be the Ricci tensor. Coupling this to the stress-energy tensor as

$$R_{\mu\nu} = \kappa T_{\mu\nu} \tag{4.18}$$

where here κ is a constant, this can be taken to be a first guess at a suitable set of field equations. The only issue is that of conservation of energy, i.e.

$$\nabla^\mu T_{\mu\nu} = 0 \tag{4.19}$$

where $R_{\mu\nu}$ does not satisfy this divergence, as can be recalled from Eq. (3.76). However, there is a known tensor which is conserved, namely the Einstein tensor, Eq. (3.77)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{4.20}$$

which allows the proposition to be made

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{4.21}$$

for the field equation for the metric. The only task left is to determine the unknown constant κ, and this will be done by comparing to the weak field, slow moving, static limit. Since the rest energy, $\rho = T_{00}$, is the dominant term in this tensor in this limit, all that is needed is to consider the $\mu, \nu = 0$ component of Eq. (4.21). Recalling Eq. (4.9) and Eq. (4.10), Eq. (4.20) can be shown to be

$$R_{00} = \frac{1}{2}\kappa T_{00} \quad . \tag{4.22}$$

To understand this in terms of the metric tensor recall $R_{00} = R^{\lambda}_{0\lambda 0}$, where here only the spatial indices need be considered since $R^{0}_{000} = 0$. Therefore in general we have

$$R^{i}_{0j0} = \partial_j \Gamma^{i}_{00} - \partial_0 \Gamma^{i}_{j0} + \Gamma^{i}_{j\lambda}\Gamma^{\lambda}_{00} - \Gamma^{i}_{0\lambda}\Gamma^{\lambda}_{j0} \quad . \tag{4.23}$$

Here the second time derivative can be neglected since a static field is assumed, and the third and fourth terms can also be neglected since they are of order two in the metric perturbation. Thus what is left is

$$\begin{aligned} R_{00} &= R^{i}_{0i0} \\ &= \partial_i \Gamma^{i}_{00} \\ &= \partial_i \left(\frac{1}{2} g^{i\lambda} (\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \right) \\ &= -\frac{1}{2} \eta^{ij} \partial_i \partial j h_{00} \\ &= -\frac{1}{2} \nabla^2 h_{00} \end{aligned} \tag{4.24}$$

which can be equated to

$$\nabla^2 h_{00} = -\kappa T_{00} \quad . \tag{4.25}$$

This happens to be exactly Eq. (4.17) if the identification is made $\kappa = 8\pi G$ and the full Einstein field equations are arrived at

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \quad . \tag{4.26}$$

These are a set of second order partial differential equations for the metric tensor.

CHAPTER 5

4-DIMENSIONAL SCHWARZSCHILD BLACK HOLES

Having flushed out the necessary philosophical and mathematical details regarding the framework used in the general theory of relativity, it is time now to delve into the interesting objects known as black holes. Black holes are objects which have a gravitational field strength so large that not even light can climb out once past a certain distance, the event horizon.

Within the framework of classical general relativity of four spacetime dimensions, there essentially is one general solution for an asymptotically flat, stationary black hole called the Kerr-Newman metric. This metric describes the geometry of a rotating, electrically charged source. There are other subsets of this general metric, namely the Schwarzschild metric, the Reissner-Nordstrom metric, and the Kerr metric which take into account various symmetries of the geometry, of which the Schwarzschild case will be investigated in this chapter. Beyond this however, motivated by string theory and the plethora of extra dimensions which come with it, higher dimensional Schwarzschild black holes and their properties will be investigated in later chapters.

5.1 The Schwarzschild Metric

What follows is a summarization from [12] [13]. The Schwarzschild metric describes the geometry outside any spherically symmetric vacuum spacetime (black holes will come soon enough). In terms of Eq. (4.26), the Einstein equation becomes $R_{\mu\nu} = 0$. The symmetries of the S^2 sphere which carry over for the Schwarzschild metric have with it three Killing vectors, (T^1, T^2, T^3) that satisfy the following commutation relations

$$[T^1, T^2] = T^3 \tag{5.1a}$$

$$[T^2, T^3] = T^1 \tag{5.1b}$$

$$[T^3, T^1] = T^2 \tag{5.1c}$$

which incidentally are the relations of SO(3), the group of rotations in 3D. Frobenius's theorem [13] can be used to argue that if some vector fields do not commute but rather have a closed commutator, then the integral curves of the vector fields can be sewn together as submanifolds of the larger manifold they are defined in. Based off Eq. (5.1), 2-spheres are formed, and almost every point in the manifold will exist on one of these, aside from the origin of a particular coordinate system; a foliation of the manifold.

To determine an appropriate coordinate scheme to impose on this n-dimensional spacetime foliation, introduce u^i and v^k, where u^i are a set of m coordinate functions on the S^2 submanifolds (i runs from 1 to n) and v^k are a set of n-m coordinate functions which identifies which submanifold is occupied (k runs from 1 to n-m). This scheme coordinatizes the entire space. Thus it can be shown the metric describing the entire manifold takes the form

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= g_{IK}(v) dv^I dv^K + f(v)\gamma_{ik} du^i du^k \end{aligned} \tag{5.2}$$

where here γ_{ik} is the metric of the submanifold. This assures the tangent vectors $\frac{\partial}{\partial v^I}$ remain orthogonal to the submanifolds.

Since 2-spheres describe the symmetric case in hand, the coordinates which can describe the submanifold are simply the angular terms in the spherical coordinate scheme, namely (θ, ϕ), while the metric on the surface of these spheres looks to be

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \quad . \tag{5.3}$$

Obviously two more coordinates are needed to fully describe the 4D spacetime, and coupled with the form of Eq. (5.2) the metric can be made to appear as (after appropriate coordinate transformations)

$$ds^2 = a(t,r)dt^2 + b(t,r)dr^2 + r^2 d\Omega^2 \quad . \tag{5.4}$$

Comparing the above expression to the known Minkowski metric, $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$, the function $a(t,r)$ can be made to be negative. Finally, using the functional freedom for the coefficients of the forms in Eq. (5.4), the general metric can be defined as

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\Omega^2 \quad . \tag{5.5}$$

This is as far as one can go in terms of a spherically symmetric, general metric. The next step is to turn the crank by placing this metric into the Einstein equations and solving for the coefficients, see [17] [18] [19]. The final result is the full Schwarzschild metric which describes the geometry outside a static, spherically symmetric source

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{5.6}$$

in geometrized units. Here M serves as a parameter which can be interpreted as the mass at large distances away. As $M \to 0$, the Minkowski line element is recovered and as $r \to \infty$ again the Minkowski result is obtained; this property is asymptotic flatness, a property which will be important for more complicated geometries describing black holes. It turns out that the above metric is unique as a spherically symmetric vacuum solution, and this fact is known from Birkhoff's theorem. Lastly before moving on to the behavior of Schwarzschild geodesics, it can be observed there are two singular points in Eq. (5.6), $r = 0$ and $r = 2M$. Since the coefficients of the metric are coordinate-dependent quantities, it makes sense to construct a scalar which may be able to shed some light on the nature of the singularities. One such scalar is the Kretschmann scalar which reads as [20]

$$R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = \frac{48M^2}{r^6} \quad . \tag{5.7}$$

Here the coordinate value $r = 2M$ clearly does not present any issue, but the value $r = 0$ does indeed represent a manifold singularity of infinite curvature.

5.2 Kruskal - Szekeres Coordinates and Penrose Diagrams

By observing the light cone structure near the $r = 2M$ distance, its nature is somewhat unmasked. This is done by considering radial null curves with constant θ and ϕ assumed

$$ds^2 = 0 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 \tag{5.8}$$

and rearranging yields

$$\frac{dt}{dr} = \pm\left(1 - \frac{2M}{r}\right)^{-1} \quad . \tag{5.9}$$

This represents the slope of the light cones in the (t, r) slice of spacetime. As one approaches the $r = 2M$ radius, the light cone boundary squeezes up to an infinite slope. As it turns out this property is an illusion caused by the particular coordinate system used, and what follows is a change in the coordinates used to unravel the nature of this boundary.

A qualitative picture of the Schwarzschild manifold, M_4, is expressed in Fig. (5.1) for the submanifold M_2 in terms of (r, ϕ) coordinates [21].

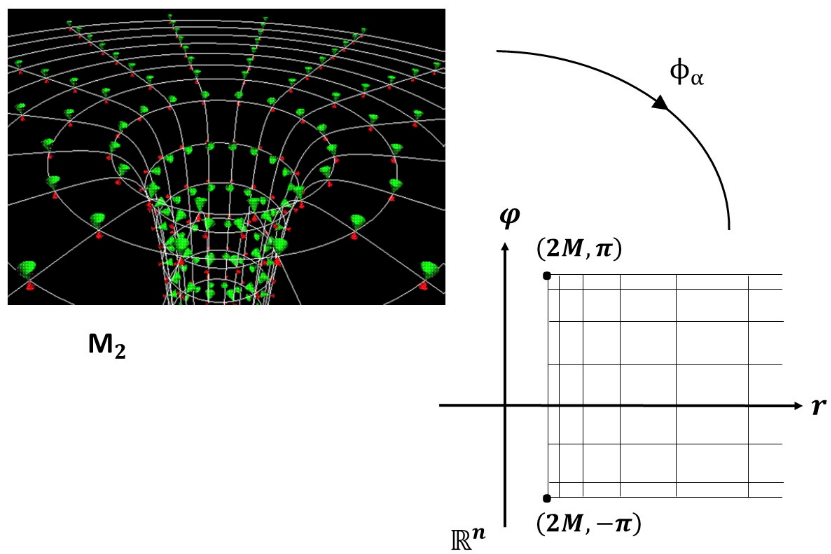


Fig. 5.1. A mapping of the 2D submanifold M_2 of the Schwarzschild spacetime. Observe how the light cones tip over once the threshold of the well has been traversed.

If the metric in Eq. (5.6) is again considered, but with again constant θ and ϕ slices, progress can be made by factoring it like so [22]

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right)\left(dt^2 - \frac{1}{(1 - \frac{2M}{r})^2} dr^2\right) \\ &= -\left(1 - \frac{2M}{r}\right)\left(dt - \frac{dr}{1 - \frac{2M}{r}}\right)\left(dt + \frac{dr}{1 - \frac{2M}{r}}\right) \end{aligned} \tag{5.10}$$

and if one makes the following coordinate definition

$$
\begin{aligned}
du &= dt - \frac{dr}{1-\frac{2M}{r}} \\
dv &= dt + \frac{dr}{1-\frac{2M}{r}}
\end{aligned}
\tag{5.11}
$$

the first equation can be subtracted from the second and integrated to yield

$$
\frac{1}{2}(v-u) = r + 2M\ln\left(\frac{r}{2M}-1\right) \quad . \tag{5.12}
$$

Notice the radius $r = 2M$ still returns a singular location, but for $r > 2M$ the coordinates u and v extend over a finite region. To remedy this issue exponentiate Eq. (5.12) like so

$$
\begin{aligned}
e^{\frac{v-u}{4M}} &= e^{\frac{r}{2M}}\left(\frac{r}{2M}-1\right) \\
&= e^{\frac{r}{2M}}\frac{r}{2M}\left(1-\frac{2M}{r}\right) \quad .
\end{aligned}
\tag{5.13}
$$

Putting this result back into Eq. (5.10) yields the metric

$$
\begin{aligned}
ds^2 &= -\frac{2M}{r}e^{-\frac{r}{2M}}e^{\frac{v-u}{4M}}dudv \\
&= -\frac{32M^3}{r}e^{-\frac{r}{2M}}\left(e^{-\frac{u}{4M}}\frac{du}{4M}\right)\left(e^{-\frac{v}{4M}}\frac{dv}{4M}\right) \\
&= -\frac{32M^3}{r}e^{-\frac{r}{2M}}dUdV
\end{aligned}
\tag{5.14}
$$

where

$$
\begin{aligned}
U &= -e^{-\frac{u}{4M}} \\
V &= e^{\frac{v}{4M}} \quad .
\end{aligned}
\tag{5.15}
$$

If the angular coordinates are added back for a full metric, one has

$$
ds^2 = -\frac{32M^3}{r}e^{-\frac{r}{2M}}dUdV + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \quad . \tag{5.16}
$$

Observation of the metric in Eq. (5.16) reveals that the distance $r = 2M$ is now perfectly behaved, therefore showing a coordinate singularity all along instead of a geometrical one. This system of coordinates (U, V, θ, ϕ) is known as Kruskal-Szekeres coordinates. The radial coordinate r can be related to the new coordinates U and V by

$$UV = -e^{-\frac{v-u}{4M}} = e^{\frac{r}{2M}}\left(1 - \frac{r}{2M}\right) \tag{5.17}$$

and yet another set of coordinates, orthogonal in nature, can be defined as

$$\begin{aligned} U &= T - X \\ V &= T + X \quad . \end{aligned} \tag{5.18}$$

The Kruskal-Szekeres coordinates (U, V, θ, ϕ) can be maximally extended over the entire Schwarzschild geometry, see Fig. (5.2). The U and V axes divide the geometry into four regions. The angular coordinates have been suppressed, therefore each point in the spacetime diagram corresponds to a 2-sphere of radius r. The shaded wedge is the original Schwarzschild region, bordered by two horizons, one past and one future. One great property of this coordinate representation is that the light cone structure is preserved from flat space, i.e. light cones are oriented with boundaries at 45 degrees. Therefore it becomes obvious the horizons at $U, V = 0$ are light-like and once crossed there is no escape since one is bound by their local light cone. Thus $r = 2M$ describes the event horizon of a black hole.

A time reversed copy region exits in the bottom quadrant of the figure from which everything must eventually escape; a white hole. The left quadrant represents another Schwarzschild region, but cannot communicate with the region on the right since faster than light speed would have to be achieved. Also, there are lines of constant radius

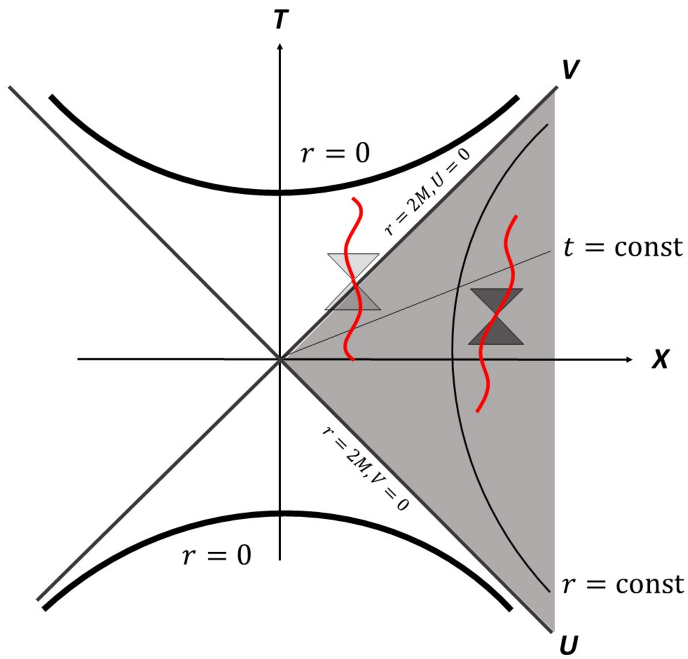


Fig. 5.2. The maximally extended Kruskal-Szekeres coordinates. The red line denotes the wordline for the respective light cone and once beyond the boundary $r = 2M$ it is not possible for a massive particle to avoid the singularity, since it is bound by its light cone to remain inside the event horizon.

r (hyperbolas) and constant time t (straight lines). Observers at a particular shell distance must accelerate in order to avoid falling into the black hole. It is interesting to note that once inside the black hole, the radial and time coordinates switch, where r is now space-like and t is now time-like, making it impossible not to continue on to smaller and smaller radial values ultimately colliding with the singularity at $r = 0$.

Lastly, there is one more transformation which is useful which brings coordinate values at infinity to finite coordinate values. The details will be suppressed here, but suffice it to say upon an inverse tangent function transformation one can create a pictorial representation, called the Penrose diagram, for the maximally extended Schwarzschild solution [13], see Fig. (5.3).

The boundaries brought from infinity in the Penrose diagram are not part of the

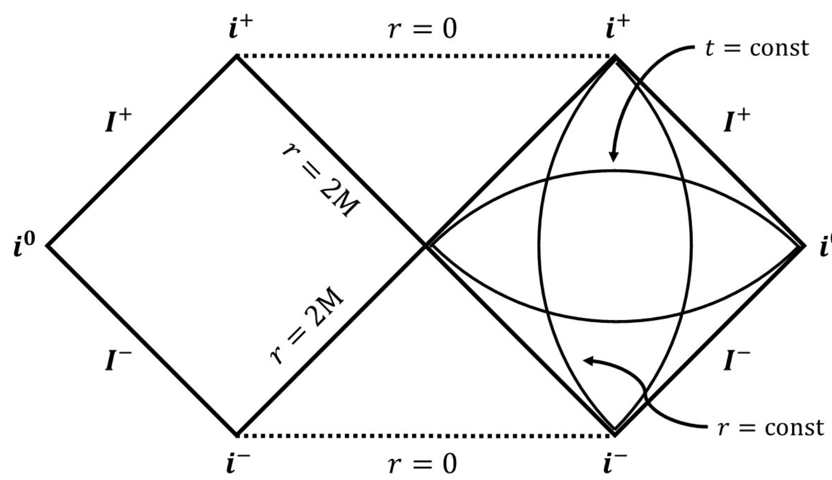


Fig. 5.3. Penrose diagram for the maximally extended Schwarzschild solution.

original spacetime, and together are referred to as conformal infinity. The conformal infinity can be divided into regions as follows

- i^+ = future time-like infinity
- i^- = past time-like infinity
- i^0 = spatial infinity
- I^+ = future null infinity
- I^- = past null infinity

Here, i^+ and i^- are separate from $r = 0$ since outside the event horizon there are an infinite number of time-like paths which do not hit the singularity.

5.3 Geodesic Equations

In order to determine the geodesic structure in the geometry of Eq. (5.6), one could grind through the brute force method of taking derivatives of the metric to find the connections, and combinations of derivatives and products of connections to arrive at the curvature tensor components, and so on and so forth. However, an alternative way is to look at the variation of the line element

$$\frac{ds^2}{d\lambda^2} = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{5.19}$$

and explicitly this reads as

$$\begin{aligned} L &= \int ds \\ &= \int \Big[-\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \Big]^{\frac{1}{2}} \quad . \end{aligned} \tag{5.20}$$

Parametrizing ds by λ yields

$$L = \int d\lambda \Big[-\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{1}{1-\frac{2M}{r}}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 \Big]^{\frac{1}{2}} \tag{5.21}$$

where here the dot denotes a derivative with respect to an arbitrary curve parameter. Now advancing with the variation set to equal zero, one can reach

$$\begin{aligned} \delta L &= \frac{1}{2}\int ds \Big[-\delta\left(\left(1-\frac{2M}{r}\right)\dot{t}^2\right) + \delta\left(\frac{1}{1-\frac{2M}{r}}\dot{r}^2\right) + \delta(r^2\dot{\theta}_1^{\,2}) + \delta(r^2\sin^2\theta\dot{\phi}^2) \Big] \\ &= 0 \end{aligned} \tag{5.22}$$

where the arbitrary curve parameter λ has been replaced with the line element differential. Evaluating the variation of each term in Eq. (5.22), recasting the variation of derivatives of coordinates as derivatives of the variation of the coordinates, and collecting terms yields

$$
\begin{aligned}
\delta L = \frac{1}{2}\int ds \Bigg[& 2\left(\left(1-\frac{2M}{r}\right)\ddot{t} + \frac{2M}{r^2}\dot{t}\dot{r}\right)\delta(t) \\
& + \left(-\frac{2\ddot{r}}{1-\frac{2M}{r}} + \frac{2M}{r^2}\dot{t}^2 + \frac{2M}{r^2\left(1-\frac{2M}{r}\right)^2}\dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2\right)\delta(r) \\
& + 2\left(r^2\sin\theta\cos\theta\dot{\phi}^2 - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta}\right)\delta(\theta) \\
& - 2\left(2r\dot{r}\sin^2\theta\dot{\phi} + 2r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + r^2\sin^2\theta\ddot{\phi}\right)\delta(\phi)\Bigg] \\
& = 0 \quad .
\end{aligned}
\tag{5.23}
$$

Since the variations $\delta(t)$, $\delta(r)$, $\delta(\theta)$, and $\delta(\phi)$ are arbitrary and can vary independently, the only way the integral can equate to zero is for the coefficients to equate to zero, yielding the geodesic equations

$$0 = \ddot{t} + \frac{2M}{r^2\left(1-\frac{2M}{r}\right)}\dot{t}\dot{r} \tag{5.24a}$$

$$0 = \ddot{r} + \frac{M}{r^2}\left(1-\frac{2M}{r}\right)\dot{t}^2 - \frac{M}{r^2\left(1-\frac{2M}{r}\right)}\dot{r}^2 - r\left(1-\frac{2M}{r}\right)\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) \tag{5.24b}$$

$$0 = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 \tag{5.24c}$$

$$0 = \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} \quad . \tag{5.24d}$$

Due to the spherical symmetry of the geometry, solutions will simply lie in an equatorial plane slicing through a great circle of the sphere. This allows initial conditions to be chosen such that $\theta = \frac{\pi}{2}$ and $\dot{\theta}(0) = 0$ and the geodesic equations become

$$0 = \ddot{t} + \frac{2M}{r^2\left(1 - \frac{2M}{r}\right)}\dot{t}\dot{r} \tag{5.25a}$$

$$0 = \ddot{r} + \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)\dot{t}^2 - \frac{M}{r^2\left(1 - \frac{2M}{r}\right)}\dot{r}^2 - r\left(1 - \frac{2M}{r}\right)\dot{\phi}^2 \tag{5.25b}$$

$$0 = \ddot{\theta} \tag{5.25c}$$

$$0 = \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} \quad . \tag{5.25d}$$

5.4 Symmetry Considerations for the Geodesic Equations and Further Analysis

For the Schwarzschild case, it is indeed possible to solve the geodesic equations outright, but in more complicated geometries, which do not contain such a high degree of symmetry, this process is greatly complicated. Therefore, hearkening back to §3.6, the general technique of utilizing Killing vectors to draw out the symmetries of the geometry shall be employed. Observing Eq.(5.6), it is obvious the line element is cyclic in the t and ϕ coordinates, thus translations $t + dt$ and $\phi + d\phi$ leave the metric invariant. The Killing vectors must then be $T = \partial_t$ and $\Phi = \partial_\phi$ with coordinate representations

$$\begin{aligned} T_\mu &= g_{\mu\nu}T^\nu = \left(-\left(1 - \frac{2M}{r}\right), 0, 0, 0\right) \\ \Phi_\mu &= g_{\mu\nu}\Phi^\nu = (0, 0, 0, r^2) \quad . \end{aligned} \tag{5.26}$$

Thus, along the geodesic there are two constants of the motion

$$L = \Phi_\mu u^\mu = r^2\dot{\phi} \tag{5.27a}$$

$$E = -T_\mu u^\mu = \left(1 - \frac{2M}{r}\right)\dot{t} \tag{5.27b}$$

where the constants L and E can be interpreted as the angular momentum and energy, respectively.

In addition, there is another constant of the motion always true for geodesics, which stems from metric compatibility, see §3.5, that looks like

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{5.28}$$

where $\epsilon = +1$ for timelike geodesics, $\epsilon = 0$ for null geodesics, and $\epsilon = -1$ for spacelike geodesics. This allows the radial equation, $\dot{r}^2$, to be expressed in terms of the energy and angular momentum, yielding three first order differential equations from the original four second order differential equations

$$\dot{\phi} = \frac{L}{r^2} \tag{5.29a}$$

$$\dot{t} = \frac{E}{1 - \frac{2M}{r}} \tag{5.29b}$$

$$\dot{r}^2 = E^2 - \left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right) \quad . \tag{5.29c}$$

Eq. (5.29c) can be recast with energy in mind to look like

$$\frac{1}{2}\dot{r}^2 + \left(\frac{1}{2}\epsilon - \frac{M\epsilon}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}\right) = \frac{1}{2}E^2 \tag{5.30}$$

where here an effective potential can be defined as [13]

$$V_{eff} = \frac{1}{2}\epsilon - \frac{M\epsilon}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} \quad . \tag{5.31}$$

The above two equations are akin to a classical particle of energy $\frac{1}{2}E^2$ moving in a 1D potential V_{eff}. A plot of the effective potential as a function of the radial coordinate for timelike geodesics can be seen in Fig. (5.4).

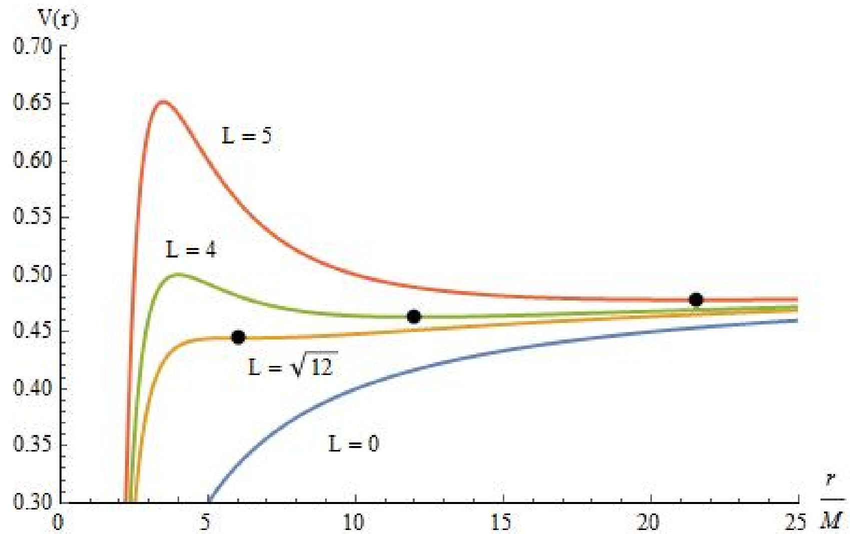


Fig. 5.4. A plot of the timelike effective potential expressed in Eq. (5.31) for unit mass as angular momentum values of $L = 0M$, $L = \sqrt{12}M$, $L = 4M$, $L = 5M$. The dots mark the stable circular orbit positions.

Much about the nature of possible orbits can be understood just from the qualitative nature of the effective potential. The particle will travel in the potential until it hits a turning point, in which case the $\dot{r}$ contribution vanishes at this location and the effective potential equals the total energy. Depending on the type of orbit, there could be two turning points (ellipse), only one turning point (unbound), or one in which the turning point is constant (circular orbit). Circular orbits occur when the effective potential is a minimum or maximum, and mathematically this is when

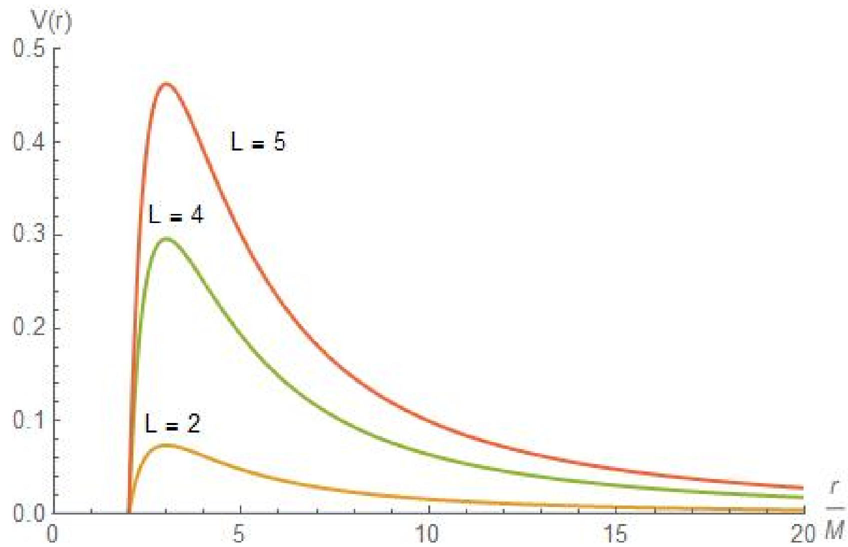


Fig. 5.5. A plot of the lightlike effective potential expressed in Eq. (5.31) for angular momentum values of $L = 2$, $L = 4M$, $L = 5M$.

$$\frac{d}{dr}V_{eff} = 0 \tag{5.32}$$

which yields

$$\epsilon M r^2 - L^2 r + 3ML^2 = 0 \quad . \tag{5.33}$$

Circular orbits are stable if they are a minimum of the potential and unstable if they are a maximum. Considering light, in which $\epsilon = 0$, one can see the only circular orbit possible is at

$$r = 3M \tag{5.34}$$

and this can be seen in Fig. (5.5) as an unstable equilibrium point before the potential dips to zero at $r = 2M$. For massive particles, in which $\epsilon = 1$, the circular orbits are at

$$r_{1,2} = \frac{L^2 \pm \sqrt{L^4 - 12M^2L^2}}{2M} \tag{5.35}$$

and coincide when the discriminant equates to zero as

$$L = \sqrt{12}M \tag{5.36}$$

which gives for the radius

$$r = \frac{L^2}{2M} = \frac{(\sqrt{12}M)^2}{2M} = 6M \quad . \tag{5.37}$$

Thus, $r = 6M$ is the smallest possible radius for a circular orbit an object with mass can have. Therefore, just from an analysis of the effective potential function, it has been found that within the Schwarzshild geometry there are stable circular orbits at $r > 6M$, and unstable circular orbits at $3M < r < 6M$.

5.5 Solving the Radial Geodesic Equation $r(\phi)$

Since the topic of interest is the geodesic structure in the Schwarzschild geometry, which has been simplified to motion in a equatorial plane, the problem boils down to solving for $r(\phi)$. This can be done simply as

$$\frac{dr}{d\lambda} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda} = \sqrt{E^2 - \left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right)} \tag{5.38}$$

and using Eq. (5.29a) the result is shown to be

$$\frac{dr}{d\phi} = \sqrt{\frac{E^2 - \epsilon}{L^2}r^4 + \frac{2M\epsilon}{L^2}r^3 - r^2 + 2Mr} = \sqrt{f(r)} \quad . \tag{5.39}$$

The above differential equation has a solution in the form of an elliptic integral. To proceed, represent the quartic function $f(r)$ in its most general form

$$f(r) = a_0 r^4 + a_1 r^3 + a_2 r^2 + a_3 r + a_4 \tag{5.40}$$

and let r_1 (a turning point as it turns out) be a zero of the function $f(r_1)$. Then the solution to Eq. (5.39) can be expressed as [23]

$$r(\phi) = r_1 + \frac{f'(r_1)}{4\wp(\phi; g_2, g_3) + \frac{1}{6}f''(r_1)} \tag{5.41}$$

where $\wp(\phi; g_2, g_3)$ is the Weierstrass $\wp$-function.

The invariants g_2 and g_3, based on the analysis in Appendix A1, therefore must be

$$g_2 = a_0 a_4 - 4a_1 a_3 + 3a_2^2 \tag{5.42a}$$

$$g_3 = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 - a_1^2 a_4 \tag{5.42b}$$

where Eq. (5.39) implies $a_4 = 0$. Eq. (5.41) is not yet a valid solution because it contains too many constants: the invariants g_2, g_3 and $f'(r)$, $f''(r)$ both depend on E and L, but in addition the zero r_1 appears. Now use r_1 and a second zero r_2 as constants of integration instead of E and L. In order to write the turning point constants in terms of the energy and angular momentum let us write $f(r)$ in the form

$$\begin{aligned} f(r) &= a_0 r(r - r_1)(r - r_2)(r - r_3) \\ &= a_0 r^4 - a_0(r_1 + r_2 + r_3)r^3 + a_0(r_1 r_2 + r_1 r_3 + r_2 r_3)r^2 - a_0 r_1 r_2 r_3 r \end{aligned} \tag{5.43}$$

and comparing this form to the form in Eq. (5.39) and Eq. (5.40) to find expressions for the coefficients, we have

$$a_0 = \frac{E^2 - \epsilon}{L^2} \tag{5.44a}$$

$$4a_1 = -a_0(r_1 + r_2 + r_3) = \frac{2M\epsilon}{L^2} \tag{5.44b}$$

$$6a_2 = a_0(r_1r_2 + r_1r_3 + r_2r_3) = -1 \tag{5.44c}$$

$$4a_3 = -a_0r_1r_2r_3 = 2M \quad . \tag{5.44d}$$

Using Eq. (5.44b) and Eq. (5.44c), an expression for a_0 is known, and one can arrive at

$$\frac{2M\epsilon}{L^2} = \frac{r_1 + r_2 + r_3}{r_1r_2 + r_1r_3 + r_2r_3} \tag{5.45}$$

and with help from Eq. (5.44a) one can find

$$E^2 - \epsilon = -\frac{2M\epsilon}{r_1 + r_2 + r_3} \quad . \tag{5.46}$$

The zero r_3 can be made to look like, with the help of Eq. (5.44c) and Eq. (5.44d)

$$r_3 = \frac{2Mr_1r_2}{r_1r_2 - 2M(r_1 + r_2)} \quad . \tag{5.47}$$

Now everything can be expressed in terms of the two zeros (turning points) r_1, r_2. The invariants g_2, g_3, given by Eq. (5.42a) and Eq. (5.42b), now look to be

$$g_2 = \frac{1}{12} - \left(\frac{2M}{4}\right)\left(\frac{r_1 + r_2 + r_3}{r_1r_2 + r_1r_3 + r_2r_3}\right) \tag{5.48a}$$

$$g_3 = -\frac{1}{48}\left(\frac{r_1 + r_2 + r_3}{r_1r_2 + r_1r_3 + r_2r_3}\right) + \frac{(2M)^2}{16}\frac{1}{r_1r_2 + r_1r_3 + r_2r_3} + \frac{1}{216} \quad . \tag{5.48b}$$

The derivative terms in the solution of Eq. (5.41) can now be found, starting with Eq. (5.43)

$$f(r) = -\frac{r(r-r_1)(r-r_2)(r-r_3)}{r_1r_2 + r_1r_3 + r_2r_3} \tag{5.49}$$

with first derivative

$$f'(r) = \frac{r_1r_2(r_3-2r) + r_1r(-2r_3+3r) + r(-2r_2r_3 + 3r_2r + 3r_3r - 4r^2)}{r_2r_3 + r_1(r_2+r_3)} \tag{5.50}$$

evaluated at $r = r_1$ yielding

$$f'(r = r_1) = -\frac{r_1(r_1-r_2)(r_1-r_3)}{r_2r_3 + r_1(r_2+r_3)} \quad . \tag{5.51}$$

The second derivative term can be found as

$$f''(r) = -\frac{2(r_2(r_3-3r) + r_1(r_2+r_3-3r) - 3r(r_3-2r))}{r_2r_3 + r_1(r_2+r_3)} \tag{5.52}$$

evaluated at $r = r_1$ yielding

$$f''(r = r_1) = \frac{-6r_1^2 - 2r_2r_3 + 4r_1(r_2+r_3)}{r_2r_3 + r_1(r_2+r_3)} \quad . \tag{5.53}$$

With these substitutions, the results from Eq. (5.41) give all possible geodesics in the form $r = r(\phi; r_1, r_2)$.

Since $r = r(\phi; r_1, r_2)$ is an elliptic function of ϕ, it must be doubly periodic with half periods ω_1 and ω_2 and these half periods depend upon the three roots of Eq. (5.43). It is not necessary to solve this equation because the solutions e_j (where $j = 1, 2, 3$) can be obtained from the roots $0, r_1, r_2, r_3$ of the quartic $f(r) = 0$ above. To see this, transform $f(r)$ to Weierstrass's normal form by a change of variables $r = \frac{1}{x}$ so that from Eq. (5.40) one arrives at

$$f(r) = a_0\left(\frac{1}{x^4}\right) + 4a_1\left(\frac{1}{x^3}\right) + 6a_2\left(\frac{1}{x^2}\right) + 4a_3\left(\frac{1}{x}\right)$$
$$= \frac{1}{x^4}(a_0 + 4a_1x + 6a_2x^2 + 4a_3x^3) \quad . \tag{5.54}$$

Introducing the term

$$\frac{1}{r} = x = \frac{1}{a_3}\left(e - \frac{a_2}{2}\right) \tag{5.55}$$

one arrives at

$$f(r) = \frac{a_3^2}{\left(e - \frac{a_2}{2}\right)^4}(4e^3 - g_2e - g_3) \tag{5.56}$$

with g_2, g_3 given by Eq. (5.48). Therefore the roots of $f(r)$ can be expressed as

$$e_j = \frac{a_3}{r_j} + \frac{a_2}{2} = \frac{r_s}{4r_j} - \frac{1}{12} \tag{5.57}$$

where here is introduced the Schwarzschild radius r_s. Eq. (5.56) with real coefficients has either three real roots or one real root and two complex conjugated roots. For a polynomial of degree three

$$f(x) = ax^3 + bx^2 + cx + d \tag{5.58}$$

the discriminant goes as

$$\Delta_3 = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd \tag{5.59}$$

so for the case at hand one has

$$\Delta_3 = 4^2(g_2^3 - 27g_3^2) \quad . \tag{5.60}$$

Three real roots corresponds to the discriminant being positive and one real, two complex conjugated roots corresponds to the discriminant being negative. The physically interesting case is when there are three real roots.

5.6 Bound Orbits

This sections proceeds with the analysis of bound orbits with relativistic corrections. For bound orbits, one has two turning points $r_2 > r_1 > 0$ and $e_1 > 0 > e_2 > e_3$ given by [24]

$$e_1 = \frac{r_s}{4r_3} - \frac{1}{12} = \frac{1}{6} - \frac{r_s}{4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{5.61a}$$

$$e_2 = \frac{r_s}{4r_1} - \frac{1}{12} \tag{5.61b}$$

$$e_3 = \frac{r_s}{4r_2} - \frac{1}{12} \quad . \tag{5.61c}$$

The real half period ω of the $\mathscr{P}$-function is given by [24]

$$\omega = \int_{e_1}^{\infty} \frac{1}{\sqrt{4t^3 - g_2 t - g_3}} dt = \frac{K(k^2)}{\sqrt{e_1 - e_3}} \tag{5.62}$$

where $K(k^2)$ is the complete elliptic integral of the first kind with parameter

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} = r_s \frac{r_2 - r_1}{r_1 r_2 - r_s(2r_1 + r_2)} = \frac{r_s}{r_1}\frac{2\varepsilon}{1+\varepsilon}\left(1 - r_s \frac{3-\varepsilon}{1+\varepsilon}\right)^{-1} \quad . \tag{5.63}$$

The parameter ε is the eccentricity of the orbit defined by

$$\frac{r_1}{r_2} = \frac{1-\varepsilon}{1+\varepsilon} \quad . \tag{5.64}$$

The goal of this section is to map out the geodesics found bound particles, and therefore what follows is the calculation of the post-Einsteinian correction to the orbital precession. The parameter k^2 in Eq. (5.63), if assumed small, can be expanded in the numerator of Eq. (5.62) as [24]

$$K(k^2) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots\right] \quad . \tag{5.65}$$

The denominator in Eq. (5.62) can be expressed as

$$\frac{1}{\sqrt{e_1 - e_3}} = 2\left[1 + \frac{r_s}{2}\frac{(2r_1 + r_2)}{r_1 r_2} + \frac{3}{8}r_s^2\left(\frac{2r_1 + r_2}{r_1 r_2}\right)^2 + O(r_s^3)\right] \tag{5.66}$$

and therefore Eq. (5.62) can be explicitly expressed as

$$\begin{aligned}\omega = \frac{\pi}{2}&\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots\right] \\ &\cdot 2\left[1 + \frac{r_s}{2}\frac{(2r_1 + r_2)}{r_1 r_2} \quad + \frac{3}{8}r_s^2\left(\frac{2r_1 + r_2}{r_1 r_2}\right)^2 + O(r_s^3)\right]\end{aligned} \tag{5.67}$$

and reduced to

$$\omega = \pi\left[1 + \frac{3}{2}\frac{r_s}{r_1}\left(\frac{1}{1+\varepsilon}\right) + \frac{3}{18}\left(\frac{r_s}{r_1}\right)^2 \frac{18+\varepsilon^2}{(1+\varepsilon)^2} + O(r_s^3)\right] \quad . \tag{5.68}$$

The perihelion precession is given by $\Delta\phi = 2(\omega - \pi)$. In Eq. (5.68), to order of r_s is Einstein's result and to order r_s^2 is the post-Einsteinian correction.

Relativistic corrections to Eq. (5.41) can be found by expressing the $\mathscr{P}$-function in terms of theta functions [25] [26], see Appendix C.

$$\vartheta_1(z,q) = 2q^{\frac{1}{4}}(\sin(z) - q^2\sin(3z) + q^6\sin(5z) - \cdots) \tag{5.69a}$$

$$\vartheta_2(z,q) = 2q^{\frac{1}{4}}(\cos(z) + q^2\cos(3z) + q^6\cos(5z) + \cdots) \tag{5.69b}$$

$$\vartheta_3(z,q) = 1 + 2q(\cos(2z) + q^3\cos(4z) + q^8\cos(6z) + \cdots) \tag{5.69c}$$

$$\vartheta_4(z,q) = 1 - 2q(\cos(2z) - q^3\cos(4z) + q^8\cos(6z) - \cdots) \tag{5.69d}$$

where here q is called the Nome [24]

$$q = \frac{k^2}{16} + 8\left(\frac{k^2}{16}\right)^2 + \cdots \quad . \tag{5.70}$$

Since k^2 was assumed to be small, the above series gives the natural expansion in powers of the Schwarzschild radius r_s. The $\mathscr{P}$-function in terms of theta functions is [24]

$$\begin{aligned}\mathscr{P}(\phi) &= e_2 + \frac{\pi^2}{4\omega^2}\left(\frac{\vartheta_1'(0)\vartheta_3(\varphi)}{\vartheta_3(0)\vartheta_1(\varphi)}\right)^2 \\ &= \frac{r_s}{4r_1} - \frac{1}{12} + \frac{\pi^2}{2\omega^2}\frac{1}{\sin^2\varphi}(1 + 4q(\cos^2\varphi - 1) + O(q^2))\end{aligned} \tag{5.71}$$

where

$$\varphi = \frac{\pi}{2\omega}\phi \tag{5.72}$$

The first and second derivative functions can be found to be

$$f'(r = r_1) = 2r_1\left(\frac{\varepsilon}{1+\varepsilon} - \frac{r_s}{r_1}\frac{3\varepsilon + \varepsilon^2}{(1+\varepsilon)^2}\right) \tag{5.73a}$$

$$f''(r = r_1) = -2\frac{1-5\varepsilon}{1+\varepsilon} + 6\frac{r_s}{r_1}\frac{1 - 4\varepsilon - \varepsilon^2}{(1+\varepsilon)^2} \tag{5.73b}$$

which yields for the $\mathscr{P}$-function

$$\begin{aligned}\mathscr{P}(\varphi) - \frac{f''(r_1)}{24} =& \frac{1 + \varepsilon - 2\varepsilon\sin^2\varphi}{4(1+\varepsilon)\sin^2\varphi} \\ &\cdot \left[1 + \frac{r_s}{r_1}\frac{1}{1 + \varepsilon\cos(2\varphi)}\left(-3 - \frac{\varepsilon}{2}(1 - \cos\varphi) + 2\varepsilon\left(\frac{3+\varepsilon}{1+\varepsilon}\right)\sin^2\varphi\right)\right] \\ &+ O\left(\frac{r_s}{r_1}\right)^2 \quad .\end{aligned} \tag{5.74}$$

Substituting this result into the original solution Eq. (5.41) yields the desired orbit to $O(r_S)$

$$\begin{aligned}\frac{r(\phi)}{r_1} =& \frac{1+\varepsilon}{1+\varepsilon\cos(2\varphi)} + \frac{r_s}{r_1}\varepsilon\frac{2\sin^2\varphi}{1+\varepsilon\cos(2\varphi)} \\ &\cdot\left(\frac{1}{1+\varepsilon\cos(2\varphi)}\left[3+\frac{\varepsilon}{2}(1-\cos\varphi)-2\varepsilon\left(\frac{3+\varepsilon}{1+\varepsilon}\right)\sin^2\varphi\right]-\frac{3+\varepsilon}{1+\varepsilon}\right) \quad .\end{aligned} \tag{5.75}$$

If the roots r_1 and r_2 coincide then Eq. (5.51) equates to zero and one obtains circular orbits. If the roots r_1, r_2, and r_3 coincide then $r_3 = 3r_s$ from Eq. (5.47), which is the innermost circular orbit possible, see Fig. (5.6).

5.7 Unbound Orbits and Null Orbits

Just for completeness, the following brief unbound and null orbit analyses are included. For the case of unbound orbits, the only physical point is the point of closest approach r_1, where the zero r_2 is negative from $\varepsilon \geq 0$ yielding

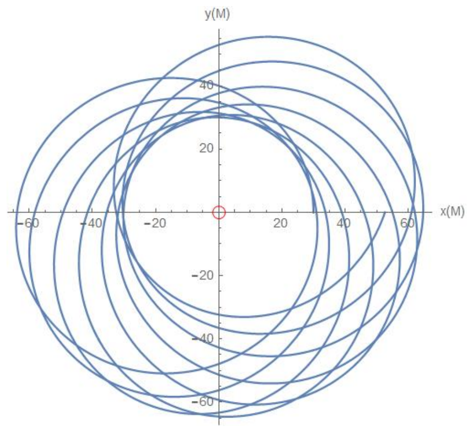
$$\frac{1+\varepsilon}{1-\varepsilon}r_1 = r_2 < 0 < r_3 < r_1 \quad . \tag{5.76}$$

From Eq. (5.39), one can calculate the orbital asymptote at infinity as

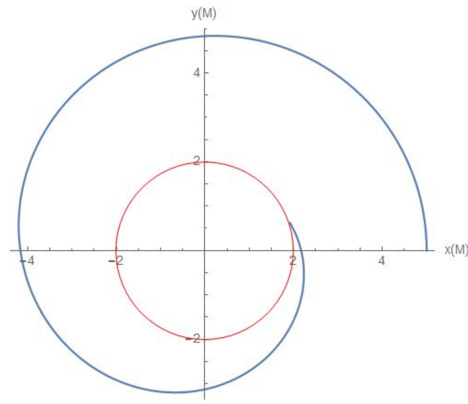
$$\phi_\infty = \int_{r_1}^{\infty} \frac{dr}{\sqrt{f(r)}} \quad . \tag{5.77}$$

The above equation is an elliptic integral and transforming to Legendre's normal form one arrives at a incomplete elliptic integral of the first kind

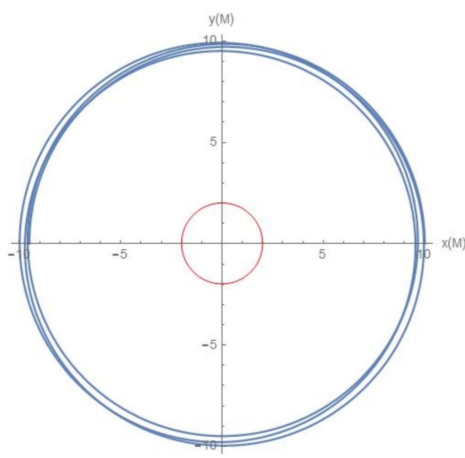
$$\phi_\infty = \frac{\mu}{\sqrt{a_0}} \int_0^{\Phi_2} \frac{d\Phi}{\sqrt{1-k^2\sin^2\Phi}} \tag{5.78}$$



(a) $r_1 = 30M$, $r_2 = 70M$



(b) $r_1 = 5M$, $r_2 = 2.5M$



(c) $r_1 = 10M$, $r_2 = 9.5M$

Fig. 5.6. Geodesic bound orbit structure for particles around Schwarzschild event horizon radius seen in red, based on Eq. (5.87). **(a)** Stable geodesic path for test particle which starts at $r_1 = 30M$, where precession is seen. **(b)** Unstable geodesic path for test particle which starts at $r_1 = 5M$, where path collides with horizon. **(c)** Stable geodesic path for test particle which starts at $r_1 = 10M$, where orbit approaches a circular one.

with transformations

$$\sin^2\Phi = \frac{r_3 - r_2}{r_1 - r_2}\frac{r - r_1}{r - r_3} \tag{5.79a}$$

$$\sin^2\Phi_2 = \frac{r_3 - r_2}{r_1 - r_2} \tag{5.79b}$$

and parameters

$$k^2 = \frac{r_3}{r_1}\frac{r_1 - r_2}{r_3 - r_2} \tag{5.80a}$$

$$\mu = \frac{2}{\sqrt{r_1(r_3 - r_2)}} \quad . \tag{5.80b}$$

It can therefore be shown the asymptotic behavior of ϕ becomes

$$\phi_\infty = 2\Phi_2 + \frac{r_s}{r_1}\left(\frac{3}{\varepsilon + 1}\Phi_2 - \frac{\varepsilon}{2(\varepsilon + 1)}\sin(2\Phi_2)\right) \quad . \tag{5.81}$$

For the case of null orbits, $\epsilon = 0$ in Eq. (5.39) and the resulting quartic function becomes more manageable. The resulting asymptotic behavior becomes

$$\phi_\infty = \frac{\pi}{2} + \delta + \left(\frac{3}{4} + \frac{3}{16}\pi\right)\delta^2 \tag{5.82}$$

with the parameter δ given by

$$\delta = \frac{r_s E}{L} \quad . \tag{5.83}$$

CHAPTER 6

D-DIMENSIONAL SCHWARZSCHILD BLACK HOLES

Given the recent (the last few decades) attempts to unify general relativity with quantum field theory, higher dimensional spacetimes have become a more common article in theoretical physics. Due to this, the nature of black holes in these higher dimensional theories can be important for the understanding of non-perturbative effects in quantum gravtiy.

In a general sense, higher dimensional black hole systems are used as testing grounds for higher dimensional theories such as string theory and Kaluza-Klein theory. Tangherlini's motivation for generalizing the Schwarzschild and Reissner-Nordstrom black holes to higher dimensions was to attempt to come to a deeper understanding as to why the observed spacetime is 3+1 dimensions.

6.1 The Schwarzschild-Tangherlini Metric

As a generalization of the 4D case, the higher dimensional case is constructed with two key concepts in mind: the first being asymptotically flat spaces are considered which implies a topology of S^{D-2} at infinity, and secondly coordinates are chosen such that the metric takes the form $g_{\mu\nu} = \eta_{\mu\nu} + O(\frac{1}{r^{D-3}})$. This will allow the mass and angular momentum to be defined by comparison to a weakly gravitating and non-relativistic case, similar to how the coupling constant for the Einstein equation was found.

The weakly gravitating metric can be expressed as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{6.1}$$

with $|h_{\mu\nu}| << 1$. Considering the harmonic gauge condition

$$\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h^{\alpha}_{\alpha}\right)_{,\nu} = 0 \quad . \tag{6.2}$$

Einstein's equation can be made to look like

$$\nabla^2 h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu} T\right) \quad . \tag{6.3}$$

Utilizing the Green's function for the generalized dimensional Laplacian, one can solve for $h_{\mu\nu}$ and expand the solution in the asymptotic region far from the source to identify the following terms to leading order [27]

$$h_{00} = \frac{16\pi G}{(D-2)\Omega_{D-2}} \frac{M}{r^{D-3}} \tag{6.4a}$$

$$h_{ij} = \frac{16\pi G}{(D-2)(D-3)\Omega_{D-2}} \frac{M}{r^{D-3}} \delta_{ij} \tag{6.4b}$$

$$h_{0i} = -\frac{8\pi G}{\Omega_{D-2}} \frac{x^k}{r^{D-1}} J^{ki} \quad . \tag{6.4c}$$

These equations can be used to define the mass and angular momentum in the source's center of mass frame.

In the most general sense, the generalization of the Schwarzschild geometry to higher dimensions reads as

$$ds^2 = -f^2 dt^2 + g^2 dr^2 + r^2 d\Omega^2_{D-2} \tag{6.5}$$

with r the radial coordinate, $d\Omega_{D-2}$ the line element of the unit $(D-2)$-sphere, and f and g functions of r only. $R_{\mu\nu} = 0$ implies

$$f = g^{-1} = \left(1 - \frac{\mu}{r^{D-3}}\right)^{\frac{1}{2}} \quad . \tag{6.6}$$

To avoid naked singularities, prohibited by the cosmic censorship conjecture, one assumes $\mu > 0$ and can use Eq. (6.4a) to find

$$\mu = \frac{16\pi G}{(D-2)\Omega_{D-2}} M \tag{6.7}$$

which yields the generalized dimensional version of the Schwarzschild 4D geometry discovered by Tangherlini in 1963 [7]. The metric for this geometry is very reminiscent of its earlier counterpart and goes as

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{D-2} \tag{6.8}$$

where here $d\Omega^2_{D-2}$ is [7] [28]

$$d\Omega^2_{D-2} = d\theta_1^2 + \sum_{j=2}^{D-2}\left(\prod_{k=1}^{j-1}\sin^2\theta_k\right)d\theta_j^2 \tag{6.9}$$

as compared to Eq. (5.3) for the 4D case, and $f(r)$ is given by [29]

$$\begin{aligned} f(r) &= 1 - \frac{\mu}{r^{D-3}} \\ &= 1 - \frac{16\pi G_D M}{(D-2)A_{D-2}r^{D-3}} \quad . \end{aligned} \tag{6.10}$$

The gravitational coupling constant has been reinstated due to its dependence on the dimensionality D and A_{D-2} is the surface area of the unit S^{D-2} sphere given as [29]

$$A_{D-2} = \frac{2\pi^{\frac{(D-1)}{2}}}{\Gamma(\frac{D-1}{2})} \quad . \tag{6.11}$$

The relationship between the surface area of the S^{D-2} sphere as a function of the dimension can be seen in Fig. (6.1).

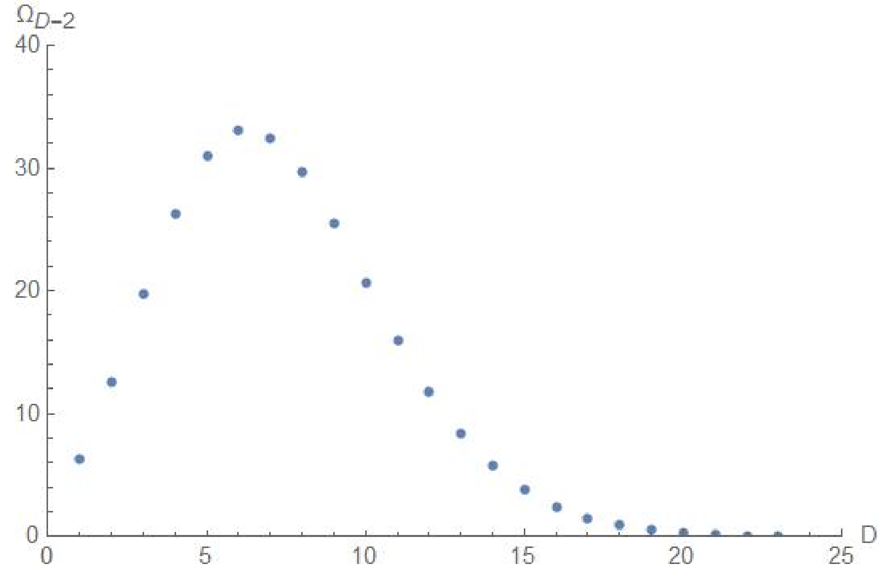


Fig. 6.1. A relationship between the surface area of the unit S^{D-2}-sphere and the dimensionality of the spacetime.

The horizon structure of the S-T black hole can be found, as before, by looking at where the radial coordinate becomes singular

$$g_{rr}^{-1} = 1 - \frac{16\pi G_D M}{(D-2)A_{D-2} r^{D-3}} = 0 \tag{6.12}$$

where the horizon radius goes as [29]

$$r_{S-T} = \left(\frac{16\pi G_D M}{(D-2)A_{D-2}}\right)^{\frac{1}{D-3}} \quad . \tag{6.13}$$

It is interesting to see how the horizon radius r depends upon the dimensionality D. This can be seen by plotting this relationship seen in Fig. (6.2). Strangely enough, it appears the horizon radius initially tends to shrink as the dimensionality is increased beyond $D = 4$, with a minimum radius of approximately $r = \frac{3}{4}M$ at dimension $D = 7.25695$, and then increases indefinitely, reaching $r = 2M$ again around dimension seventy, with no convergence as $D \to \infty$.

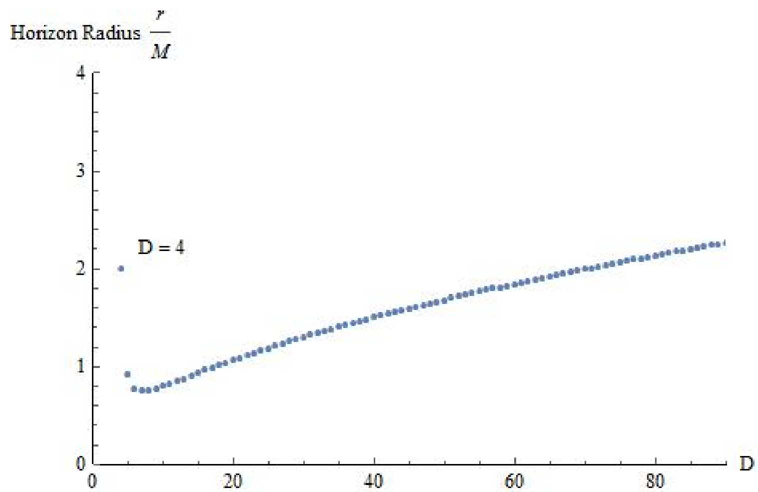


Fig. 6.2. A relationship between the radius of the event horizon for the generalized Schwarzschild black hole and the dimensionality of the spacetime. Here on can see the correlation between Fig. (6.1) and the behavior of the horizon radius.

6.2 Generalized Kruskal-Szekeres Coordinates

Observing the light cone structure near the horizon radius r_{S-T} will allow this coordinate singularity to be transformed away. Consider null curves with all angles fixed

$$ds^2 = 0 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 \tag{6.14}$$

which yields the light cone slope in the generalized (t, r) spacetime

$$\frac{dt}{dr} = \pm\left(1 - \frac{\mu}{r^{D-3}}\right)^{-1} \quad . \tag{6.15}$$

As r_{S-T} is approached, the light cone squeezes up to infinite slope, but the actual radius at which this occurs depends on the dimension considered.

Consider again the metric in Eq. (6.8), for constant angular slices, and factor like so

$$
\begin{aligned}
ds^2 &= -f(r)\left(dt^2 - \frac{1}{f(r)^2 dr^2}\right) \\
&= -f(r)\left(dt - \frac{dr}{f(r)}\right)\left(dt + \frac{dr}{f(r)}\right) \quad .
\end{aligned}
\tag{6.16}
$$

Making the following coordinate definitions

$$
\begin{aligned}
du &= dt - \frac{dr}{f(r)} \\
dv &= dt + \frac{dr}{f(r)}
\end{aligned}
\tag{6.17}
$$

and subtracting the first equation from the second and integrating yields

$$
\frac{1}{2}(v-u) = \int \frac{dr}{f(r)} \quad . \tag{6.18}
$$

Now depending on the dimension one is considering, the solution to the integral in Eq. (6.18) will vary. Table (6.1) encapsulates some of the dimensions with corresponding solutions to the integral.

As one can probably see, the difficulty in finding a suitable coordinate system to transform away the coordinate singularity at the event horizon becomes increasingly difficult with higher dimensions, see [30].

6.3 Generalized Geodesic Equations

In order to determine the geodesic structure in the geometry of Eq. (6.8), the same method will be employed here that was used for the 4D case. Again, the variation in the line element goes as

$$
\begin{aligned}
L &= \int ds \\
&= \int \left[-f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta_1^2 + r^2 \sum_{j=2}^{D-2} \left(\prod_{k=1}^{j-1} \sin^2\theta_k \right) d\theta_j^2 \right]^{\frac{1}{2}} \quad .
\end{aligned}
\tag{6.19}
$$

Dimension	Solution
D = 4	$r+\mu\ln(r-\mu)$
D = 5	$r-\mu^{\frac{1}{2}}\tanh^{-1}(\mu^{-\frac{1}{2}}r)$
D = 6	$r-\frac{1}{6}\mu^{\frac{1}{3}}\Big(2\sqrt{3}\tan^{-1}(\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}\mu^{-\frac{1}{3}}r)-2\ln\big(-r+\mu^{\frac{1}{3}}\big)$ $+\ln\big(r^2+\mu^{\frac{1}{3}}r+\mu^{\frac{2}{3}}\big)\Big)$
D = 7	$r-\frac{1}{4}\mu^{\frac{1}{4}}\Big(2\tan^{-1}(\mu^{-\frac{1}{4}}r)-\ln(-r+\mu^{\frac{1}{4}})+\ln(r+\mu^{\frac{1}{4}})\Big)$
D = 8	$r+\frac{1}{20}\mu^{\frac{1}{5}}\Big(-2\sqrt{10+2\sqrt{5}}\tan^{-1}\big(\frac{4r+(1-\sqrt{5})\mu^{\frac{1}{5}}}{\sqrt{10+2\sqrt{5}}\mu^{\frac{1}{5}}}\big)$ $-2\sqrt{10-2\sqrt{5}}\tan^{-1}\big(\frac{4r+(1+\sqrt{5})\mu^{\frac{1}{5}}}{\sqrt{10-2\sqrt{5}}\mu^{\frac{1}{5}}}\big)+4\ln\big(-r+\mu^{\frac{1}{5}}\big)$ $+(-1+\sqrt{5})\ln\big(r^2-\frac{1}{2}(-1+\sqrt{5})\mu^{\frac{1}{5}}r+\mu^{\frac{2}{5}}\big)$ $-(1+\sqrt{5})\ln\big(r^2+\frac{1}{2}(1+\sqrt{5})\mu^{\frac{1}{5}}r+\mu^{\frac{2}{5}}\big)\Big)$
D = 9	$r+\frac{1}{12}\mu^{\frac{1}{6}}\Big(2\sqrt{3}\tan^{-1}\big(\frac{1}{\sqrt{3}}-\frac{2r}{\sqrt{3}\mu^{\frac{1}{6}}}\big)-2\sqrt{3}\tan^{-1}\big(\frac{1}{\sqrt{3}}+\frac{2r}{\sqrt{3}\mu^{\frac{1}{6}}}\big)$ $+2\ln\big(-r+\mu^{\frac{1}{6}}\big)-2\ln\big(r+\mu^{\frac{1}{6}}\big)$ $+\ln\big(r^2-\mu^{\frac{1}{6}}r+\mu^{\frac{1}{3}}\big)-\ln\big(r^2+\mu^{\frac{1}{6}}r+\mu^{\frac{1}{3}}\big)$
$\vdots$	$\vdots$

Table 1. Table of solutions to Eq. (6.18) RHS integral with corresponding dimension.

Parametrizing ds by λ yields

$$L=\int d\lambda\Big[-f(r)\dot{t}^2+\frac{1}{f(r)}\dot{r}^2+r^2\dot{\theta}_1^{\,2}+r^2\sum_{j=2}^{D-2}\Big(\prod_{k=1}^{j-1}\sin^2\theta_k\Big)\dot{\theta}_j^{\,2}\Big]^{\frac{1}{2}}\quad . \tag{6.20}$$

Now advancing with the variation, one can reach

$$\delta L = \frac{1}{2}\int ds\Bigg[-\delta\Bigg(\Big(1-\frac{\mu}{r^{D-3}}\Big)\dot{t}^2\Bigg) + \delta\Bigg(\frac{1}{(1-\frac{\mu}{r^{D-3}})}\dot{r}^2\Bigg) + \delta\big(r^2\dot{\theta_1}^2\big) + \delta\Bigg(r^2\sum_{j=2}^{D-2}\Big(\prod_{k=1}^{j-1}\sin^2\theta_k\Big)\dot{\theta_j}^2\Bigg)\Bigg]^{\frac{1}{2}} \tag{6.21}$$

where the arbitrary curve parameter λ has been replaced with the line element differential. Evaluating the variation of each term in Eq. (6.21), recasting the variation of derivatives of coordinates as derivatives of the variation of the coordinates, and collecting terms yields

$$\begin{aligned}\delta L = \frac{1}{2}\int ds\Bigg[&2\Bigg(\Big(1-\frac{\mu}{r^{D-3}}\Big)\ddot{t} + \frac{(D-3)\mu}{r^{D-2}}\dot{t}\dot{r}\Bigg)\delta(t)\\ &+\Bigg(-\frac{2\ddot{r}}{(1-\frac{\mu}{r^{D-3}})} + \frac{(D-3)\mu}{r^{D-2}}\dot{t}^2 + \frac{3(D-3)\mu}{r^{D-2}(1-\frac{\mu}{r^{D-3}})^2}\dot{r}^2\\ &\qquad + 2r\dot{\theta}_1^2 + 2r\sum_{j=2}^{D-2}\Big(\prod_{k=1}^{j-1}\sin^2\theta_k\Big)\dot{\theta_j}^2\Bigg)\delta(r)\\ &-2\Big(2r\dot{r}\dot{\theta}_1 + r^2\ddot{\theta}_1\Big)\delta(\theta_1) - 4r\dot{r}\sum_{j=2}^{D-2}\Big(\prod_{k=1}^{j-1}\sin^2\theta_k\Big)\dot{\theta}_j\delta(\theta_j)\\ &-2r^2\sum_{j=2}^{D-2}\Big(\prod_{k=1}^{j-1}2\sin\theta_k\cos\theta_k\dot{\theta}_k\Big)\dot{\theta}_j\delta(\theta_j) - 2r^2\sum_{j=2}^{D-2}\Big(\prod_{k=1}^{j-1}\sin^2\theta_k\Big)\ddot{\theta}_j\delta(\theta_j)\Bigg]\\ &= 0 \quad .\end{aligned} \tag{6.22}$$

Since the variations are arbitrary and can vary independently, the only way the integral can equate to zero is for the coefficients to equate to zero, yielding the geodesic equations

$$
\begin{aligned}
0 &= \ddot{t} + \frac{\mu(D-3)}{r^{D-2}(1-\frac{\mu}{r^{D-3}})}\dot{t}\dot{r} && (6.23a) \\
0 &= \ddot{r} + \frac{\mu(D-3)}{2r^{D-2}}\left(1-\frac{\mu}{r^{D-3}}\right)\dot{t}^2 - \frac{1}{2}\frac{\mu(D-3)}{r^{D-2}(1-\frac{\mu}{r^{D-3}})}\dot{r}^2 && (6.23b) \\
&\qquad - r\left(1-\frac{\mu}{r^{D-3}}\right)\left(\dot{\theta}_1^2 + \sum_{j=2}^{D-2}\left(\prod_{k=1}^{j-1}\sin^2\theta_k\right)\dot{\theta}_j^2\right) \\
0 &= \ddot{\theta}_1 + \frac{2}{r}\dot{r}\dot{\theta}_1 - \sum_{j=2}^{D-2}\left(\prod_{k=1}^{j-1}\sin\theta_k\cos\theta_k\right)\dot{\theta}_j^2 && (6.23c) \\
&\vdots \\
0 &= \ddot{\theta}_j + \frac{2}{r}\dot{\theta}_j + \sum_{j=2}^{D-2}\left(\prod_{k=1}^{j-1}2\cot\theta_k\dot{\theta}_k\right)\dot{\theta}_j \quad . && (6.23d)
\end{aligned}
$$

One can check that these generalized geodesic equations indeed reduce to the 4D case when $D = 4$.

Due to the spherical symmetry of the geometry, solutions will simply lie in an equatorial plane slicing through a great circle of the unit S^{D-2} sphere. This allows initial conditions to be chosen such that $\theta_1 = \theta_2 = \theta_3 = \cdots = \theta_{D-3} = \frac{\pi}{2}$ and the geodesic equations become

$$0 = \ddot{t} + \frac{\mu(D-3)}{r^{D-2}(1-\frac{\mu}{r^{D-3}})}\dot{t}\dot{r} \tag{6.24a}$$

$$0 = \ddot{r} + \frac{\mu(D-3)}{2r^{D-2}}\left(1-\frac{\mu}{r^{D-3}}\right)\dot{t}^2 - \frac{1}{2}\frac{\mu(D-3)}{r^{D-2}(1-\frac{\mu}{r^{D-3}})}\dot{r}^2 - r\left(1-\frac{\mu}{r^{D-3}}\right)\dot{\theta}_{D-2}^2 \tag{6.24b}$$

$$0 = \ddot{\theta}_1 \tag{6.24c}$$

$$\vdots$$

$$0 = \ddot{\theta}_{D-2} + \frac{2}{r}\dot{\theta}_{D-2} \quad . \tag{6.24d}$$

6.4 Symmetry Considerations for the Generalized Geodesic Equations and Further Analysis

In order to follow the same logic as the 4D case, the Killing vectors will be sought after. Observing Eq.(6.8) and Eq. (6.9), the line element is cyclic in the t and θ_{D-2} coordinates, thus translations $t+dt$ and $\theta_{D-2}+d\theta_{D-2}$ leave the metric invariant. The Killing vectors must then be $T = \partial_t$ and $\Phi = \partial_{\theta_{D-2}}$ with coordinate representations

$$\begin{aligned} T_\mu = g_{\mu\nu}T^\nu &= \left(-\left(1-\frac{\mu}{r^{D-3}}\right), 0, 0, 0, \ldots, 0\right) \\ \Phi_\mu = g_{\mu\nu}\Phi^\nu &= \left(0, 0, 0, 0, \ldots, r^2\prod_{k=1}^{j-1}\sin^2\theta_k\right) \end{aligned} \tag{6.25}$$

and since an equatorial slice is assumed then the Killing vector Φ_μ becomes

$$\Phi_\mu = (0, 0, 0, 0, \ldots, r^2) \quad . \tag{6.26}$$

Along the geodesic there must be two constants of the motion

$$L = \Phi_\mu u^\mu = r^2 \dot{\theta}_{D-2} \tag{6.27a}$$

$$E = -T_\mu u^\mu = \left(1 - \frac{\mu}{r^{D-3}}\right)\dot{t} \tag{6.27b}$$

where the constants L and E can still be interpreted as the angular momentum and energy, respectively. Utilizing Eq. (5.28) again, three first order equations can be arrived at

$$\dot{\theta}_{D-2} = \frac{L}{r^2} \tag{6.28a}$$

$$\dot{t} = \frac{E}{1 - \frac{\mu}{r^{D-3}}} \tag{6.28b}$$

$$\dot{r}^2 = E^2 - \left(1 - \frac{\mu}{r^{D-3}}\right)\left(\frac{L^2}{r^2} + \epsilon\right) \quad . \tag{6.28c}$$

Eq. (6.28c) can be recast with energy in mind to look like

$$\frac{1}{2}\dot{r}^2 + \left(\frac{1}{2}\epsilon - \frac{\mu\epsilon}{2r^{D-3}} + \frac{L^2}{2r^2} - \frac{\mu L^2}{2r^{D-1}}\right) = \frac{1}{2}E^2 \tag{6.29}$$

where here an effective potential can be defined as

$$\begin{aligned} V_{eff} &= \frac{1}{2}\epsilon - \frac{\mu\epsilon}{2r^{D-3}} + \frac{L^2}{2r^2} - \frac{\mu L^2}{2r^{D-1}} \\ &= \frac{1}{2}\epsilon - \frac{8\pi G_D M\epsilon}{(D-2)\Omega_{D-2} r^{D-3}} + \frac{L^2}{2r^2} - \frac{8\pi G_D M L^2}{(D-2)\Omega_{D-2} r^{D-1}} \quad . \end{aligned} \tag{6.30}$$

A plot of the effective potential as a function of the radial coordinate for timelike geodesics can be seen in Fig. (6.3) while the effective potential for lightlike geodesics can be seen in Fig. (6.4).

Here it can be observed that the position of where the effective potential orig-

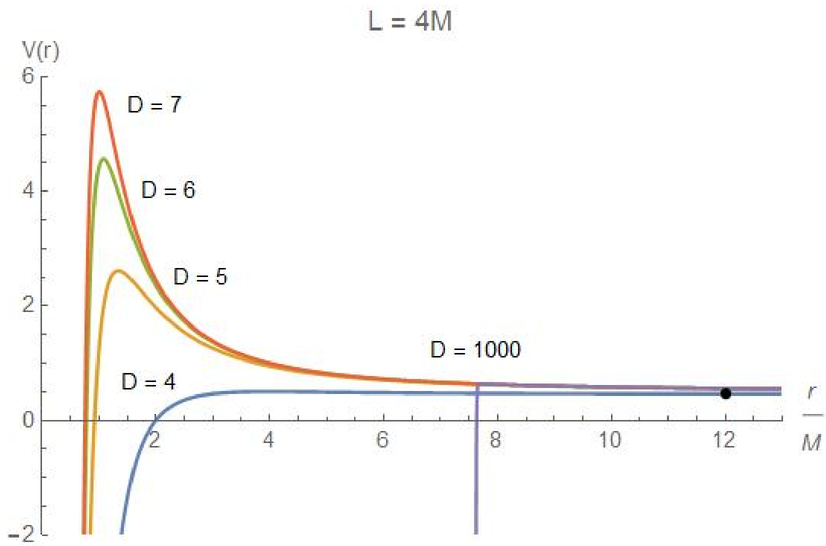


Fig. 6.3. A plot of the timelike effective potential expressed in Eq. (6.30) for dimensions $D = 4, 5, 6, 7, 1000$ with unit mass and angular momentum value $L = 4M$. The black dot represents the stable circular orbit for the $D = 4$ case, and is the only bound orbit possible; all higher dimensions have terminating orbits.

inates initially decreases as the dimension increases, and in fact this continues to a minimum horizon radius only to increase again, in accordance with Fig. (6.2).

To observe the turning points as a function of dimension, simply differentiate V_{eff} and set the result equal to zero, which yields

$$\mu\epsilon(D-3)r^{5-D} + \mu L^2(D-1)r^{3-D} - 2L^2 = 0 \quad . \tag{6.31}$$

One can check that for the 4D timelike case, Eq. (5.35) is indeed arrived at after solving the second order polynomial. For the 5D case, one can arrive at the turning points

$$r_{1,2} = \pm 4L\sqrt{\frac{GM}{3\pi L^2 - 8\epsilon GM}} \tag{6.32}$$

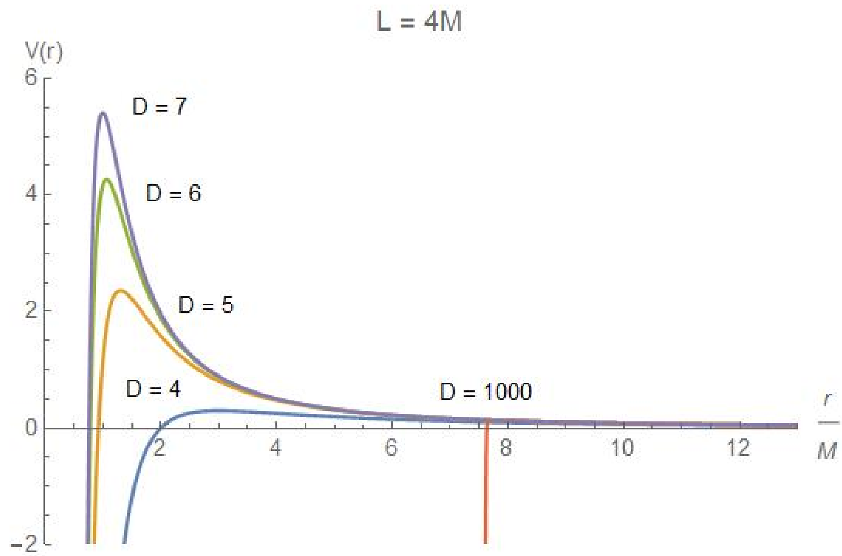


Fig. 6.4. A plot of the lightlike effective potential expressed in Eq. (6.30) for dimensions $D = 4, 5, 6, 7, 1000$ with angular momentum value $L = 4M$. The unstable circular orbits for photons can be seen to decrease with dimension initially then increase in accordance with the Fig. (6.1).

where it is obvious there is only one physical turning point, therefore no bound orbits, only terminating orbits. The 6D case and beyond yields general results for the turning points which are quite cumbersome, therefore their calculation is left to the reader, but suffice it to say a common theme amidst them all is only one physical turning point. Thus, one can surmise that beyond four dimensions, there exist no periodic bound orbits, and only terminating bound, escape, or terminating escape orbits.

6.5 Solving the Generalized Radial Geodesic Equation $r(\phi)$

The next step is to attempt a generalization of the radial differential equation and see what types of solutions exist for $r(\theta_{D-2})$. One can expand Eq. (6.21c) as

$$\frac{dr}{d\lambda} = \frac{dr}{d\theta_{D-2}}\frac{d\theta_{D-2}}{d\lambda} = \sqrt{E^2 - \left(1 - \frac{\mu}{r^{D-3}}\right)\left(\frac{L^2}{r^2} + \epsilon\right)} \tag{6.33}$$

and with the help of Eq. (6.21a) it is easily shown

$$\frac{dr}{d\theta_{D-2}} = \sqrt{\frac{E^2-\epsilon}{L^2}r^4 - r^2 + \mu r^{5-D} + \frac{\mu\epsilon}{L^2}r^{7-D}} \quad . \tag{6.34}$$

There does not exist a general solution to this differential equation, rather one must specify a dimension and seek the particular type of function which the solution can be expressed in the form of. With a substitution of $u = \frac{\mu^{\frac{1}{D-3}}}{r}$, Eq. (6.34) can be made to look like [25]

$$\left(\frac{du}{d\theta_{D-2}}\right)^2 = u^{D-1} + \lambda u^{D-3} - u^2 + \lambda(E^2 - \epsilon) = P_{D-1}(u) \tag{6.35}$$

where the parameter $\lambda = \frac{\mu^{\frac{2}{D-3}}}{L^2}$, and for what follows the θ_{D-2} angle will just be taken to be ϕ.

For the case of $D = 5$ the substitution $u = \frac{1}{x} + l$, with l a zero of P_4, turns Eq. (6.35) into

$$\left(\frac{dx}{d\phi}\right)^2 = b_3x^3 + b_2x^2 + b_1x^1 + b_0x^0 \tag{6.36}$$

with coefficients

$$b_3 = 4l^3 + 2l(\lambda - 1) \tag{6.37a}$$

$$b_2 = 6l^2 + \lambda - 1 \tag{6.37b}$$

$$b_1 = 4l \tag{6.37c}$$

$$b_0 = 1 \quad . \tag{6.37d}$$

Another substitution of $x = \frac{1}{b_3}(4y - \frac{b_2}{3})$ puts Eq. (6.36) in the form of Eq. (A.6), which can be solved in terms of the Weierstrass elliptic functions.

For the case of $D = 6$, Eq. (6.35) looks like

$$\left(\frac{du}{d\phi}\right)^2 = u^5 + \lambda u^3 - u^2 + \lambda(E^2 - \epsilon) = P_5(u) \tag{6.38}$$

where the physical angle is now given by

$$\phi - \phi_0 = \int_{u_0}^{u} \frac{du'}{\sqrt{P_5(u')}} = \int_{u_0}^{u} dz_1 \tag{6.39}$$

and the solution to the radial geodesic equation is given by [25]

$$r(\phi) = \frac{\mu^{\frac{1}{3}}}{u(\phi)} = -\mu^{\frac{1}{3}} \frac{\sigma_2(\phi_{\vec{\Theta},6})}{\sigma_1(\phi_{\vec{\Theta},6})} \tag{6.40}$$

where

$$\vec{\phi}_{\Theta,6} = \begin{pmatrix} \phi - \phi_0' \\ \phi_1 \end{pmatrix} \quad . \tag{6.41}$$

Here, ϕ_1 is chosen such that $(2\omega)^{-1}\vec{\phi}_{\Theta,6}$ is an element of the theta divisor $\Theta_{\vec{K}_\infty}$ and $\phi_0' = \phi_0 + \int_{u_0}^{\infty} dz_1$ depends only on initial values u_0 and ϕ_0, see Appendix B and C.

For the $D = 7$ case, one has a polynomial P_6 of degree six on the RHS of Eq. (6.35). After another substitution of $u = \frac{1}{x} + l$, where l is a zero of P_6, one obtains the differential equation

$$\left(x\frac{dx}{d\phi}\right)^2 = b_5 x^5 + \cdots + b_0 x^0 \tag{6.42}$$

for appropriate constants $b_5, b_4, \ldots, b_0$. The solution to this differential equation is [25]

$$r(\phi) = \frac{\mu^{\frac{1}{4}}}{u(\phi)} = -\mu^{\frac{1}{4}} \frac{\sigma_2(\phi_{\vec{\Theta},7})}{\sigma_1(\phi_{\vec{\Theta},7})} \tag{6.43}$$

where

$$\vec{\phi}_{\Theta,7} = \begin{pmatrix} \phi_1 \\ \\ \phi - \phi_0^{'} \end{pmatrix} \quad . \tag{6.44}$$

Again, ϕ_1 is chosen such that $(2\omega)^{-1}\vec{\phi}_{\Theta,7}$ is an element of the theta divisor $\Theta_{\vec{K}_\infty}$ and $\phi_0^{'} = \phi_0 + \int_{u_0}^{\infty} dz_2$, see Appendix B and C.

The explicit analysis ends here, but one could push forward to higher dimensions, particularly dimensions nine and eleven, in a similar fashion as described above. Dimensions eight, ten, and twelve and beyond do not have analytical solutions to date.

CHAPTER 7

CONCLUSION

A significant method for exploring the spacetime geometry given by a specific metric is to determine the geodesics particles would travel in the geometry. In this thesis, the properties regarding the Schwarzschild geometry, as an exterior solution to the Einstein vacuum equation, have been explored in addition to the geodesics which arise from solving the geodesic equation for the Schwarzschild metric. The general structure of the orbits was then discussed, with an approximate solution to the radial equation being found. A higher dimensional generalization was then arrived at, discovered by Tangherlini in 1963, and solutions to the geodesic equation in higher dimensions were explored.

In four dimensions, the spacetime geometry allows for periodic bound, terminating bound, escape, and terminating escape orbits. A substantial amount of this information could be extrapolated from the qualitative analysis of the effective potential behavior. The solution to the radial geodesic equation was found using the Weierstrass $\mathscr{P}$-function, which is a periodic function of elliptic type defined on a Riemannian manifold.

In higher dimensions, the generalized Schwarzschild solution only allows for terminating bound, escape, or terminating escape orbits. This can be seen from only one physical turning point arising from the radial differential equation polynomial and also from a qualitative analysis of the generalized effective potential. Also, it was observed the horizon radius shrinks as the dimension increases and reaches a minimum around dimension seven, increasing indefinitely with larger dimensions. This is simply an artifact of the geometry itself. Dimensions four, five, and seven are solvable in terms of elliptic functions. Dimensions seven, nine, and eleven are solvable in terms of hyperelliptic functions. Dimensions eight, ten, and twelve and beyond do not have any analytical solutions known.

Appendix A

THE WEIERSTRASS $\mathscr{P}$-FUNCTION

The Weierstrass $\mathscr{P}$-function is defined as [31]

$$\mathscr{P}(z) = \frac{1}{z^2} + \sum_{\substack{n,m \in Z \\ n^2+m^2 \neq 0}} \left(\frac{1}{(z+z_{nm})^2} - \frac{1}{z_{nm}^2} \right) \tag{A.1}$$

with

$$z_{nm} = 2m\omega_1 + 2n\omega_2 \tag{A.2}$$

where ω_1 and ω_2 are the two half periods characterizing the period parallelogram, see Fig. (A.1). By construction this function is doubly periodic with fundamental periods $2\omega_1$ and $2\omega_2$.

$$\mathscr{P}(z+2\omega_1) = \mathscr{P}(z)\,, \quad \mathscr{P}(z+2\omega_2) = \mathscr{P}(z) \quad . \tag{A.3}$$

Knowing the values of the elliptic function within the fundamental period parallelogram completely determines the elliptic function, as a consequence of Eq. (A.3).

If one acquires the Laurent series for the Weierstrass function about $z = 0$, the

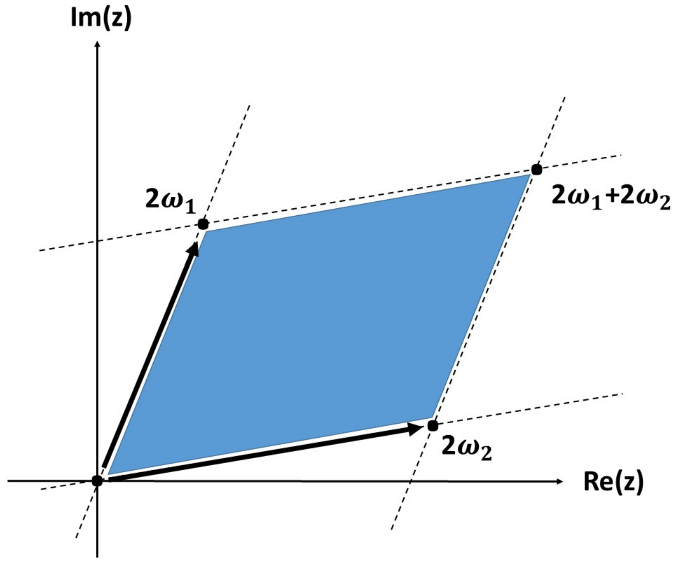


Fig. A.1. The fundamental period parallelogram shaded in blue defined by the complex numbers 0, $2\omega_1$, $2\omega_2$, and $2\omega_1 + 2\omega_2$.

expressions for the invariants g_2 and g_3 can then be determined as

$$g_2 = 60 \sum_{\substack{n,m \in Z \\ n^2+m^2 \neq 0}} \frac{1}{z_{nm}^4} \tag{A.4a}$$

$$g_3 = 140 \sum_{\substack{n,m \in Z \\ n^2+m^2 \neq 0}} \frac{1}{z_{nm}^6} \quad . \tag{A.4b}$$

In order to gain insight into the differential equation for which the Weierstrass $\mathscr{P}$-function is a solution of, one can differentiate Eq. (A.1) to arrive at

$$\mathscr{P}'(z) = -2 \sum_{n,m} \frac{1}{(z + m\omega_1 + n\omega_2)^3} \quad . \tag{A.5}$$

Using the Laurent series of $\mathscr{P}'(z)$, $(\mathscr{P}(z))^3$, and $(\mathscr{P}'(z))^2$ in the regime of $z = 0$

it can be shown the Weierstrass elliptic function obeys the differential equation

$$(\mathscr{P}'(z))^2 = 4(\mathscr{P}(z))^3 - g_2\mathscr{P}(z) - g_3 = 0 \tag{A.6}$$

and the integral formula for the Weierstrass elliptic function is

$$z = \int_{\mathscr{P}(z)}^{\infty} \frac{1}{\sqrt{4t^3 - g_2 t - g_3}} dt \quad . \tag{A.7}$$

If one defines the value of the Weierstrass function at the half-periods ω_1, ω_2, and $\omega_3 = \omega_1 + \omega_2$ as

$$e_1 = \mathscr{P}(\omega_1)\,, \quad e_2 = \mathscr{P}(\omega_3)\,, \quad e_3 = \mathscr{P}(\omega_2) \tag{A.8}$$

then one can show $\mathscr{P}$ is stationary at the half-periods

$$\mathscr{P}'(\omega_1) = \mathscr{P}'(\omega_2) = \mathscr{P}'(\omega_3) = 0 \tag{A.9}$$

and arrive at [24]

$$4e_i^3 - g_2 e_i - g_3 = 0 \quad . \tag{A.10}$$

This equation implies that e_i are the three roots of the polynomial appearing in the right hand side of the Weierstrass equation, namely

$$4t^3 - g_2 t - g_3 = 4(t - e_1)(t - e_2)(t - e_3) \tag{A.11}$$

which directly implies

$$e_1 + e_2 + e_3 = 0 \tag{A.12a}$$

$$e_1 e_2 + e_2 e_3 + e_3 e_1 = -\frac{g_2}{4} \tag{A.12b}$$

$$e_1 e_2 e_3 = \frac{g_3}{4} \quad . \tag{A.12c}$$

Appendix B

DIFFERENTIALS DEFINED ON RIEMANNIAN SURFACES

What follows is a summarization from [25]. A Riemann surface is a complex manifold with a good notion of complex-analytic functions defined on it. Let X be a compact Riemann surface of the algebraic function $x \mapsto \sqrt{P_n(x)}$ for a polynomial P_n of degree n, with representation

$$X := \{z = (x, y) \in \mathbb{C}^2 \mid y^2 = P_n(x)\} \tag{B.1}$$

or the analytic continuation of $\sqrt{P_n}$. The latter can be seen as a two chart covering of the Riemann sphere constructed with the zeros (branch points) of P_n, $e_i, i = 1, \ldots, n$. Now if one takes two copies of the Riemann sphere, for each possible value of $\sqrt{P_n}$ and cut them between every two of the branch points e_i, such that the cuts do not touch, i.e. branch cuts. A visualization of a sixth order polynomial can be seen in Fig. (B.1). The two copies are attached along the branch cuts in such a way that $\sqrt{P_n}$ with its analytic continuations is uniquely defined on the whole surface. Therefore $x \mapsto \sqrt{P_n(x)}$ is a single-valued function.

Every Riemann surface can be equipped with a homology basis $\{a_i, b_i | i = 1, \ldots, g\} \in H_1(X, \mathbb{Z})$ of closed paths, where g is the genus of the surface. The genus can be characterized as the dimension of the space of holomorphic differentials on the Riemannian surface, or the number of holes in the topology, see Fig. (B.1). If P_n has only simple zeros, the genus of surface X of $\sqrt{P_n}$ is $g = [\frac{n-1}{2}]$.

In order to construct functions on the Riemann surface, a canonical basis of the space of holomorphic differentials $\{dz_i | i = 1, \ldots, g\}$ and associated meromorphic

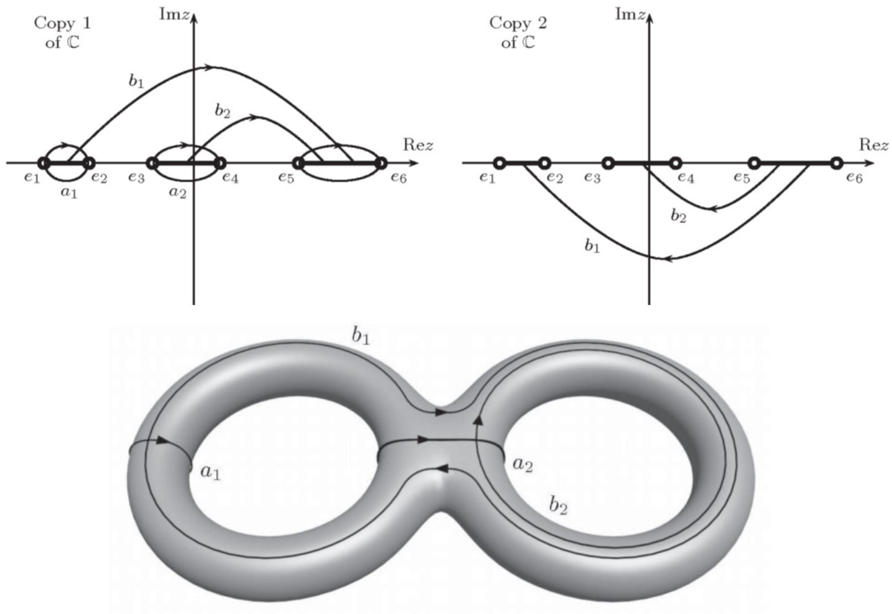


Fig. B.1. Riemann surface of genus 2 for sixth degree polynomial, with real branch points $e_1, \ldots, e_6$. The upper figure has two copies of the complex plane with closed paths giving a homology basis $\{a_i, b_i | i = 1, \ldots, g\}$. The lower figure has topology equivalent to homology basis.

differentials $\{dr_i | i = 1, \ldots, g\}$ must be defined like

$$dz_i = \frac{x^{i-1} dx}{\sqrt{P_n(x)}} \tag{B.2}$$

$$dr_i = \sum_{k=i}^{2g+1-i} (k+1-i) b_{k+1+i} \frac{x^k dx}{4\sqrt{P_n(x)}} \tag{B.3}$$

with b_i the coefficients of the polynomial $P_n(x) = \sum_{j=1}^{n} b_j x^j$. There also can be introduced period matrices $(2\omega, 2\omega^{'})$ and $(2\eta, 2\eta^{'})$ related to the homology basis

$$2\omega_{ij} = \oint_{a_j} dz_i \tag{B.4a}$$

$$2\omega^{'}_{ij} = \oint_{b_j} dz_i \tag{B.4b}$$

$$2\eta_{ij} = -\oint_{a_j} dr_i \tag{B.4c}$$

$$2\eta^{'}_{ij} = -\oint_{b_j} dr_i \quad . \tag{B.4d}$$

The differentials in Eq. (B.2) and Eq. (B.3) are defined such that the components of their period matrices satisfy the Legendre relation

$$\begin{pmatrix} \omega & \omega^{'} \\ \eta & \eta^{'} \end{pmatrix} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \begin{pmatrix} \omega & \omega^{'} \\ \eta & \eta^{'} \end{pmatrix}^T = -\frac{1}{2}\pi i \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \tag{B.5}$$

where 1_g is the $g \times g$ unit matrix.

A normalized holomorphic differential can be defined as

$$d\vec{v} = (2\omega)^{-1} d\vec{z} \tag{B.6}$$

$$d\vec{z} = \begin{pmatrix} dz_1 \\ dz_2 \\ \vdots \\ dz_g \end{pmatrix} \quad . \tag{B.7}$$

The period matrix of these differentials is given by $(1_g, \tau)$, where τ is defined as

$$\tau = \omega^{-1}\omega^{'} \quad . \tag{B.8}$$

In this thesis, the holomorphic differentials introduced are used to formulate geodesic equations of the type

$$\left(y^i \frac{dy}{dx}\right)^2 = P_n(y) \qquad , \qquad y(x_0) = y_0 \quad . \tag{B.9}$$

Appendix C

THE THETA FUNCTION

What follows is a summarization from [25]. The theta functions are a class of functions, $\vartheta : \mathbb{C}^g \to \mathbb{C}$,

$$\vartheta(\vec{z};\tau) = \sum_{\vec{m}\in\mathbb{Z}^g} e^{i\pi\vec{m}^t(\tau\vec{m}+2\vec{z})} \tag{C.1}$$

which define holomorphic functions in $\mathbb{C}^g$. The theta function is periodic with respect to the columns of 1_g and quasiperiodic with respect to the columns of τ, i.e.

$$\vartheta(\vec{z}+1_g\vec{n};\tau) = \vartheta(\vec{z};\tau) \tag{C.2a}$$

$$\vartheta(\vec{z}+\tau\vec{n};\tau) = e^{-i\pi\vec{n}'(\tau\vec{n}+2\vec{z})}\vartheta(\vec{z};\tau) \quad . \tag{C.2b}$$

The theta function will also need the characteristics $\vec{g},\vec{h} \in \frac{1}{2}\mathbb{Z}^g$ defined by

$$\begin{aligned}\vartheta[\vec{g},\vec{h}](\vec{z};\tau) &= \sum_{\vec{m}\in\mathbb{Z}^g} e^{i\pi(\vec{m}+\vec{g})'(\tau(\vec{m}+\vec{g})+2\vec{z}+2\vec{h})} \\ &= e^{i\pi\vec{g}'(\tau\vec{g}+2\vec{z}+2\vec{h})}\vartheta(\vec{z}+\tau\vec{g}+\vec{h};\tau) \quad .\end{aligned} \tag{C.3}$$

It will be important that for every $\vec{g},\vec{h}$ the set, called the theta divisor,

$$\Theta_{\tau\vec{g}+\vec{h}} = \{\vec{z}\in\mathbb{C}^g \mid \vartheta[\vec{g},\vec{h}](\vec{z};\tau) = 0\} \tag{C.4}$$

be a $(g-1)$ - dimensional subset of $Jac(X)$, where for a Riemannian surface X the Jacobian is $Jac(X) = \mathbb{C}^g/\Gamma$, with $\Gamma = \{\omega v + \omega' v' \mid v, v' \in \mathbb{Z}^g\}$ is the lattice of periods of differential $d\vec{z}$.

A useful function, related to the theta function, useful for solving the integral arising in the radial differential equation can be found by considering the theta function

$$\vartheta_e(x;\tau) = \vartheta\bigg(\int_{x_0}^{x} d\vec{v} - \vec{e};\tau\bigg) \tag{C.5}$$

with fixed $\vec{e} \in \mathbb{C}^g$. It can be shown the Riemann theta function is either identically zero or has g zeros $x_1, \ldots, x_g$ for which

$$\sum_{i=1}^{g} \int_{x_0}^{x} d\vec{v} = \vec{e} + \vec{K}_{x_0} \tag{C.6}$$

is valid with $\vec{K}_{x_0} \in \mathbb{C}^g$. $\vec{K}_{x_0}$ is the vector of Riemannian constants with respect to the base point x_0 given by

$$K_{x_0,j} = \frac{1+\tau_{jj}}{2} - \sum_{l\neq j} \oint_{a_l} \bigg(\int_{x_0}^{x} dv_j\bigg) dv_l(x) \tag{C.7}$$

where τ_{jj} is the jth diagonal element of Eq. (B.8). As $x_0 \to \infty$, $\vec{K}$ can be expressed as

$$\vec{K}_\infty = \sum_{i=1}^{g} \int_{\infty}^{e_{2i}} d\vec{v} \tag{C.8}$$

where e_{2i} is the starting point of one of the branch cuts not containing ∞ for each i, therefore $\vec{K}_\infty$ can be expressed as a linear combination of half periods.

The solution of the type of problem expressed in Eq. (B.9) can be formulated in terms of derivatives of the Kleinian sigma function $\sigma : \mathbb{C}^g \to \mathbb{C}$

$$\sigma(\vec{z}) = Ce^{-\frac{1}{2}\vec{z}^{\prime}\eta\omega^{-1}\vec{z}}\vartheta((2\omega)^{-1}\vec{z} + \vec{K}_{x_0};\tau) \tag{C.9}$$

where C is a constant. The sigma function can be used to expressed a form of a general Weierstrass function as

$$\mathscr{P}_{ij}(\vec{z}) = -\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\log\sigma(\vec{z}) = \frac{\sigma_i(\vec{z})\sigma_j(\vec{z}) - \sigma(\vec{z})\sigma_{ij}(\vec{z})}{\sigma^2(\vec{z})} \quad . \tag{C.10}$$

The solution to the general problem of the physical angle can be found in terms of the generalized Weierstrass functions. If X is a Riemannian surface of$\sqrt{P}$, with P the polynomial expressed as

$$P(x) = \sum_{i=0}^{2g+1} \lambda_i x^i \tag{C.11}$$

then the components of the solution vector $\vec{x} = (x_1, \ldots, x_g)^T$ are given by g solutions of

$$\frac{\lambda_{2g+1}}{4}x^g - \sum_{i=1}^{g} \mathscr{P}_{gi}(\vec{\phi})x^{i-1} = 0 \quad . \tag{C.12}$$

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